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ALGEBRA

FOR

SCHOOLS AND COLLEGES

ALGEBRA

FOR

SCHOOLS AND COLLEGES

BY

WILLIAM FREELAND, A.B.

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PREFACE.

THIS book is designed to meet the requirements of those students who present themselves for the maximum courses in college entrance examinations, and to furnish a course in Freshman work for students who have advanced through the subject of Quadratics only.

Great care has been taken to cover the entire ground necessary for such work.

Needless space has not been given to preliminary explanations, as it is assumed that boys and young men who begin to read Algebra have, in every case, had the proper grounding in Arithmetic.

The examples have been so selected and graded that the student will have no difficulty in proceeding step by step through the entire work.

The author has been mindful to introduce examples and problems to such an extent only as will allow the completion of each exercise as it presents itself.

The course, as here presented, can be finished by mature boys in a single year, and by boys who begin the subject at the age of fourteen or fifteen in two years.

With the exception of two or three instances, no claim to originality is made. The examples, when not original, have been selected from French and German works, and problems have been given with a view to arousing interest.

For several lucid explanations the author is indebted to the large work of Briot, and for numerous examples in the advanced chapters to the great work of Heis.

For any suggestions or correction relating to the body of the work or to the answers to examples, the author will be specially thankful.

W. FREELAND.

New York, Sept. 20, 1895.

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ALGEBRA.

CHAPTER I.

DEFINITIONS.

1. Algebra is that branch of mathematics in which letters and figures are employed to reason about the general properties of numbers.

SYMBOLS.

Algebraic symbols are as follows: —

2. The **Sign of Addition**, +, is read "plus."

Thus, $3 + 4$ equals 7; again, $a + b$ indicates that b is to be added to a.

3. The **Sign of Subtraction**, −, is read "minus."

Thus, $7 - 3$ equals 4; again, $a - b$ indicates that b is to be subtracted from a.

4. The **Sign of Multiplication**, ×, is read "times," or "multiplied by."

Thus, 3×4 equals 12; again, $a \times b$ indicates that a is to be multiplied by b.

Between letters the sign of multiplication usually is not expressed. Thus, $ab = a \times b$.

5. The **Sign of Division**, ÷, is read "divided by."

Thus, $3 \div 4$ equals $\frac{3}{4}$; again, $a \div b$ indicates that a is to be divided by b.

6. The **Sign of Equality**, $=$, is read "equals," or "is equal to."

Thus, $2 + 4 = 6$; again, $a = b$ indicates that a is equal to b.

7. The **Signs of Inequality**, $>$ and $<$, are read "is greater than" and "is less than."

Thus, $4 > 3$; again, $a < b$ indicates that a is less than b.

8. The **Signs of Aggregation**; viz.: the parentheses ( ); the vinculum $\overline{}$; the bracket [ ]; the brace { }.

These signs show that the quantities included by them are to be taken together; *i.e.* treated as a single term.

Thus, $(a + b - c) \times d$, $\overline{a + b - c} \times d$, $[a + b - c] \times d$, and $\{a + b - c\} \times d$, indicate that the result of $a + b - c$ is to be multiplied by d.

9. The **Sign of Involution** is called the "**exponent,**" and shows how many times the quantity affected is to be taken as a factor. a^2 is read "a square"; a^3 is read "a cube," or a to the third power.

Thus, $a^2 = a \times a$; $b^3 = b \times b \times b$; $(4)^3 = 4 \times 4 \times 4 = 64$.

The exponent is written at the right, above the quantity.

10. The **Sign of Evolution**, $\surd$, is called the **Radical Sign**, and indicates that the root of a quantity is to be taken.

Thus, $\sqrt{4} = 2$; again, $\sqrt{a^4} = a^2$; $\sqrt[5]{a^5} = a$.

Also, $\sqrt[3]{a + b}$ or $\sqrt[3]{}(a + b)$ indicates that the cube root of $a + b$ is to be taken.

The figure written above at the left of the radical is called the **Index**.

When the square root is to be taken, no index is expressed.

11. The **Sign of Deduction**, $\therefore$, is read "hence" or "therefore."

12. The **Sign of Continuation**, $\cdots$, is read "and so on."

13. The **Coefficient** shows how many times a quantity is to be taken.

Thus, in $3a$, 3 is the coefficient, and indicates that a is to be taken three times; again, in the quantity $3ax$, if a is known, $3a$ is the coefficient and indicates that x is to be taken $3a$ times.

14. The **Reciprocal** of a quantity is 1 divided by that quantity.

Thus the reciprocal of a is $\frac{1}{a}$.

The **Double Sign,** ±, is written before a quantity to indicate that it is to be added or subtracted.

Thus, $a \pm b$ is equivalent to $a + b$ or $a - b$.

ALGEBRAIC EXPRESSIONS.

15. A quantity which has the + or no sign before it is a **Positive Quantity.**

Thus, abc and $+xy$ are positive quantities.

16. A quantity which has the sign − before it is a **Negative Quantity.**

Thus, $-abc$ and $-xy$ are negative quantities.

17. A **Monomial** is an expression of one term.

Thus, abc and $-\frac{3y}{2x}$ are monomials.

18. A **Polynomial** is an expression of more than one term.

Thus, $a + b$, $c - d + e$, $x + 3y - 4z$, and $\frac{3}{x} + \frac{y}{2} - \frac{3}{z}$ are polynomials.

A **Term** is an expression not separated by + or −.

19. A **Binomial** is an expression of two terms.

Thus, $a + b$ and $3x + 2y$ are binomials.

20. A **Trinomial** is an expression of three terms.

Thus, $a+b+c$ and $2x+3y-4z$ are trinomials.

21. **Similar Terms** are those which have the same letters and the corresponding letters affected by the same exponents.

Thus, $2a^2$ and $-3a^2$ are similar terms.

Also, $-3x^2yz^3$ and $5x^2yz^3$ are similar terms.

22. **Homogeneous Terms** are those of the same degree.

Thus, ax^2y^3, a^6, $a^2x^2y^2$ are homogeneous.

23. The **Degree** of a term is the sum of the exponents of the literal factors.

Thus, $3xy$, x^2, and $4a^2$ are terms of the second degree.

Again, $4abcd$, $3a^2b^2$, and x^4 are of the fourth degree.

24. **Numerical Value** of an expression is obtained by giving numerical values to the letters involved.

Thus, if $a=10$, $b=20$, $c=25$, $d=0$.

The numerical value of $a+b$ is 30, of ab is 200, of $a+3b-4cd$ is 70.

25. The **Absolute Value** of a number is its value when its algebraic sign is disregarded.

Thus, the absolute value of -10 is 10.

26. The symbol which stands for a number may represent a positive or a negative number. Thus, a may be positive or negative, but in either case $-a$ will represent a number of the same absolute value with opposite sign.

Thus, if $a=-10$, $-a=+10$,

or if $a=+10$, $-a=-10$.

POSITIVE AND NEGATIVE NUMBERS.

27. Many quantities exist in two opposite states.

Latitude north and south, profit and loss, degrees of temperature above and below zero, degrees of longitude east and west, distances above and below the sea-level, hold opposite relations to each other.

The + and − signs determine these opposite relations in Algebra, in addition to indicating the operations of addition and subtraction.

If we denote any of the above relations by +, its opposite relation will be denoted by −.

Thus, degrees of temperature above zero are usually denoted by +, while degrees below zero are represented by −; north latitude may be indicated by +, then south latitude will be indicated by −; profit is calculated as +, loss is calculated as −.

This opposite relation of quantities, as considered in Algebra, gives rise to the question of subtracting a larger number from a smaller; an operation not capable of being performed in Arithmetic.

Thus, $6 - 15 = -9$, and $6 + (-9) = -3$.

The signs + and −, when used to indicate the opposite relations of quantities, are called *signs of opposition.*

28. **A factor** is one of the ~~equal~~ divisors of a number.

Thus, 3 and 4 are factors of 12, and a and b are factors of ab.

EXERCISE I.

If $a = 2$, $b = 3$, $c = 4$, $d = 5$, $e = 0$, find the value of the following expressions:

1. $a + b$.
2. $3c + 4d$.
3. $3ab + 9c - e$.
4. $abc - 6ced + 9cd$.
5. $a^2 - b^2 + c^2$.
6. $\frac{a}{b} + \frac{a^2}{b^2} - e^2$.
7. $a^2c^2d^2 - d^2e^3 + 4cd$.
8. $\frac{a^2 + d^2}{a^2 - d^2}$.
9. $\frac{a}{c + d} + c + \frac{d}{e + b}$.
10. $\frac{a^5 + b^2}{d^2} + \frac{d^2}{a^5 + b^2}$.
11. $a^2 + 2ab + b^2$.
12. $b^2 - 2bc + c^2$.

13. $a^3 + 3a^2b + 3ab^2 + b^3$.

14. $\frac{a^3 + b^3}{a + b}$.

15. $\frac{a^c + b^a - d^{2a}}{c + e^d - b}$.

16. $\frac{a}{2} - \frac{bc}{d} + \frac{ecd}{ab}$.

17. $\frac{c^2 + de + b^2c + a^3e^5}{a + c^2 - d^4 + e^6}$.

18. $\frac{d^3 - c^4 + e^{20}}{c + d + 5}$.

19. $\frac{eacd}{bcd} - \frac{a^2b^2}{b^2d^2}$.

20. $\frac{b^3 - c^3}{b^3 + c^3}$.

EXERCISE 2.

1. Express the difference of x and y.

2. John has b dollars, and James has c dollars; how many dollars have both?

3. There are x boys; each boy has y marbles; how many marbles have all of the boys?

4. Express the sum of m and n multiplied by the difference of the same quantities.

5. What is the quotient of a^3 plus c^3 divided by b^4?

6. Write 8 times the fourth power of c multiplied by 3 times the second power of d divided by the eighth power of x.

7. How many lines are there in e books, when each book has a pages, and each page b lines?

8. How many square yards in a field which is b yards wide and $a + b$ yards long?

9. If I have xy dollars, and a book cost $x + y$ dollars, how many books can I buy for my money?

10. What is the next number above x? The next number below x?

11. What are the ten consecutive numbers of which x is the fifth, and y the sixth?

CHAPTER II.

ADDITION.

29. Addition is the process of uniting two or more quantities into an equivalent expression.

The result obtained is called the **sum.**

30. Similar terms may be united into one term by addition.

31. Dissimilar terms are added together by writing one after the other with the proper algebraic sign.

Addition of Similar Terms. Like Signs.

1.	2.	3.
$2a$	$-2a$	$3abc$
$5a$	$-8a$	$4abc$
$8a$	$-6a$	$6abc$
$10a$	$-7a$	$8abc$
$25a$	$-23a$	$21abc$

32. Addition of Similar Terms. Unlike Signs.

1.	2.	3.
$+3a$	$-3ab$	$+9abcd$
$-4a$	$+3ab$	$+10abcd$
$+7a$	$+8ab$	$+12abcd$
$-5a$	$-16ab$	$-19abcd$
$+a$	$-8ab$	$+12abcd$

From the two sets of examples above it will be seen that,

33. 1st. When the signs are alike in similar terms, the sum of the coefficients prefixed to the common literal part with the given sign gives the result.

2d. When the signs are unlike, the difference between the positive and negative coefficients prefixed to the common literal part, with the sign of the greater, gives the result.

Hence the rule:

34. To add similar terms.

I. *When the signs are alike, prefix the sum of the coefficients with the given sign to the common literal part.*

II. *When the signs are unlike, find the sum of the positive and the sum of the negative coefficients, separately, and prefix the difference of these sums with the sign of the greater to the common literal part.*

35. To add polynomials.

I. *Write the quantities in separate columns, placing similar terms under each other.*

II. *Add each column separately, and connect the several results with the proper sign.*

Examples.

1.

$$\begin{array}{r} a+b \\ 2a-3b \\ 5a+6b \\ 3a-18b \\ \hline 11a-14b \end{array}$$

2.

$$\begin{array}{r} -17c+18ab \\ +15c-16ab \\ +14c-13ab \\ -c-11ab \\ \hline 11c-22ab \end{array}$$

3.

$$\begin{array}{r} a-b+2c \\ 3a-4b+6c \\ -8a-7b+8c \\ -3a-9b-16c \\ \hline -7a-21b \end{array}$$

EXERCISE 3.

Add the following:

4. $12abc,\ -11abc,\ -5abc+2abc.$
5. $18xy^2z^3,\ 13xy^2z^3,\ -21xy^2z^3,\ -40xy^2z^3.$
6. $\frac{1}{2}a^2b,\ -\frac{1}{8}a^2b,\ -\frac{1}{4}a^2b,\ +\frac{2}{6}a^2b.$
7. $a^2b-c^3d,\ 8a^2b+4c^3d,\ 23a^2b-16c^3d.$
8. $4x+3y+2z,\ 3x-4y-3z,\ 8x-3y+4z,\ x-8y-z.$
9. $a^3+a^2b+ab^2c,\ 3a^3+2a^2b+4ab^2c,\ 8a^3-6a^2b-16ab^2c.$

10. $3(a+b^2),\ 5(a+b^2),\ 6(a+b^2),\ -16(a+b^2)$.

11. $\begin{cases} 11m^2+11m-11m^3,\ 3m^2-3m^3-4m, \\ 8m^3-8m^2+14m,\ 13m-8m^2-13m^3. \end{cases}$

12. $\begin{cases} a^2+b+c-d,\ a^2+b-c,\ a^2-d, \\ b-c+4d,\ 9a^2+10b-6c+12d. \end{cases}$

Commutative Law.

No matter what the order of the terms in addition, the result will be the same.

Thus $a+b=b+a$.

Again, $5+4=4+5=9$.

This is called the commutative law of addition.

Associative Law.

Quantities may be grouped in any manner, and added together; this process is called the associative law of addition.

Thus,

$$\begin{aligned} &a+b+x+y+z \\ &=(a+b)+(x+y+z) \\ &=a+(b+x+y)+z. \end{aligned}$$

36. The **Unit of Addition** is the common letter or quantity whose coefficients are added in finding the sum of two or more quantities.

Thus, x is the unit of addition in $13x+6x=19x$.

Again, $a+b+c$ is the unit of addition in

$$3(a+b+c)+8(a+b+c)-10(a+b+c)=(a+b+c).$$

37. Dissimilar terms may have a unit of addition.

Thus, x is the unit of addition in

$$abx+bcx+edx=(ab+bc+ed)x.$$

EXERCISE 4.

Add the following:

1. ax^2, bcx^2, dcx^2, $12\,x^2$.
2. $axy + cxy$, $bxy + exy$, $m^2xy + n^2xy$.
3. $abc + efbc + xybc$, $mnbc + pqbc + edbc$.
4. $(a - b)(p + q)$, $(x - y)(p + q)$, $(m - n)(p + q)$.
5. $2\,a\sqrt{x + y}$, $- 6\,b\sqrt{x + y}$, $- 8\,c\sqrt{x + y}$.

SUBTRACTION.

38. **Subtraction** is the process of finding the difference between two quantities.

39. The student must bear in mind that the terms **addition** and **subtraction have a more general sense** in Algebra than in Arithmetic. In Arithmetic all quantities are regarded as positive; consequently the sum of two quantities will be greater than either, while the difference will be less than either.

In Algebra the sum of two quantities may be less than either, while the difference between two quantities may be greater than either.

Thus:

	ADDITION.	SUBTRACTION.
40.	$+ 9 + (+ 3) = 12$	$+ 9 - (+ 3) = 6$
	$+ 9 + (- 3) = 6$	$+ 9 - (- 3) = 12$
	$- 9 + (+ 3) = - 6$	$- 9 - (+ 3) = - 12$
	$- 9 + (- 3) = - 12$	$- 9 - (- 3) = - 6$

Using letters instead of figures for the purposes of this demonstration, we have:

$+ a + (+ 2\,a) = 3\,a$	$+ a - (+ 2\,a) = - a$
$+ a + (- 2\,a) = - a$	$+ a - (- 2\,a) = + 3\,a$
$- a + (+ 2\,a) = + a$	$- a - (+ 2\,a) = - 3\,a$
$- a + (- 2\,a) = - 3\,a$	$- a - (- 2\,a) = + a$

From the above we see that subtracting a positive number is equivalent to adding an equal negative number, and subtracting a negative number is equivalent to adding an equal positive number.

41. Hence the rule:

To subtract one quantity from another.

I. *Change the sign of each term of the subtrahend and proceed as in addition.*

Examples. — Monomials.

	1.	2.	3.	4.
From	$-4a^2x$	$8ax^2y$	$11x^3y^2z$	$-21ab^2$
Subtract	$-2a^2x$	$3ax^2y$	$-4x^3y^2z$	$4ab^2$
Remainder,	$-2a^2x$	$5ax^2y$	$15x^3y^2z$	$-25ab^2$

Examples. — Polynomials.

	5.	6.
From	$3x^2 - 4a + 9b^3$	$3abx - 6cdy + a^2b^2$
Subtract	$5x^2 + 6a - 8b^3$	$-7abx - cdy + 5a^2b^2$
Remainder,	$-2x^2 - 10a + 17b^3$	$10abx - 5cdy - 4a^2b^2$

EXERCISE 5.

7. From $x^2 - ax + x^4$ subtract $x^2 - 2ax$.

8. From $a^3 + a^2 - a + m$ subtract $3a^3 - m + 2a$.

9. From $m^2 + 3n - 4p + 6q$ subtract $3p + 5q - 3m^2 + n$.

10. From $3(p+q)$ subtract $9(p+q)$.

11. From $3ax(m^2 + n - p)$ subtract $2ax(m^2 + n - p)$.

12. From $\frac{1}{2}p + \frac{3}{2}q - \frac{4}{5}m$ subtract $\frac{1}{4}p + \frac{3}{4}q + \frac{2}{5}m$.

13. From $2a\sqrt{b} + \sqrt{c}$ subtract $5a\sqrt{b} - \sqrt{c}$.

14. From $9xy(2\sqrt{x} + 3\sqrt{y} - \sqrt{c})$
subtract $8xy(2\sqrt{x} + 3\sqrt{y} - \sqrt{c})$.

15. From $x^5 - x^3 + x - \sqrt{x}$ subtract $x^4 - 2\sqrt{x} + x^3$.

PARENTHESES.

42. From section 40 it becomes apparent that if a parenthesis is removed when a plus sign precedes it, the sign of the term within the parenthesis remains the same, also that if a parenthesis is removed when a minus sign precedes it, the sign of the term within the parenthesis is changed.

The same rule holds when the parenthesis encloses several terms.

43. Hence, in removing parentheses, we have the following rule:

I. *If a parenthesis is removed which has a plus sign before it, the terms of the parenthesis remain unchanged.*

II. *If a parenthesis is removed which has a minus sign before it, the sign of each term enclosed in the parenthesis must be changed.*

Thus, $a+(x-y)=a+x-y,$

but $a-(x-y)=a-x+y.$

44. Sometimes several parentheses occur, one enclosed within the other. In such cases each parenthesis must be treated according to the sign before it. The best method of procedure is to remove the innermost parenthesis first.

Thus:

$$3x-[-4y-\{3x-(2y+\overline{x-y})+6y\}-6x]$$
$$=3x-[-4y-\{3x-(2y+x-y)+6y\}-6x]$$
$$=3x-[-4y-\{3x-2y-x+y+6y\}-6x]$$
$$=3x-[-4y-3x+2y+x-y-6y-6x]$$
$$=3x+4y+3x-2y-x+y+6y+6x$$
$$=11x+9y.$$

EXERCISE 6.

Remove the parentheses in the following:

1. $(x+y+z)+(x-y+z)$.
2. $(2m+n)-(3m+4n)$.
3. $(3a+2b)-(3c+5a)+(3b-4c)$.
4. $1-(1-x)-(1-x^2)+(x+x^3)$.
5. $x-[+a-(b+c)-a]$.
6. $a-(b+c-\overline{x-y})$.
7. $3c-[4d+a-(c+d-x)]$.
8. $2x-\{3y+\overline{5z-4x}\}-(10z-\overline{y+3y})$.
9. $a-[b+\{c-\overline{d+3a}\}]$.
10. $3x-(4y-6z)-\{2x-(5y+4z)+2x\}$.
11. $2a-[-\{4b+(3c-\overline{a+b})+c\}-6b]$.
12. $[\{(\overline{a+b-c+d-e})\}]$.
13. $-[(a+b)+\{c+(d+a)+e\}+(e+a)-a]$.
14. $[10a-\{-(a+b)-(c+d+e)\}-6b]$.
15. $[e-\{a-(c+d)\}-b-\{-c+(d-\overline{a+b})\}+3a]$.

45. Quantities may be introduced into parentheses, if the following rules are observed:

I. *Any quantity may be put in a parenthesis and a plus sign put before it.*

II. *Any quantity may be put in a parenthesis and a minus sign before it, provided the sign of each term of the quantity placed within be changed.*

EXERCISE 7.

Express in binomials:

1. $2a - 3b - 6a + 4b$.
2. $3a - 4b + 6e + 3d - 2a + b$.
3. $3x^2 - x - y^2 + y + 3x - 4y$.

Express in trinomials:

4. $ab + bc - a - cd + ad + e$.
5. $3a - 6bc + 4de - a + ed + b$.
6. $3abx + 6bcx + a - de + abc + d$.

Express the following in trinomials, each trinomial containing an inner binomial:

7. $3a + b + c + a - b + c$.
8. $3x - 2y + z - 3y + 2x - z$.

In the following collect the coefficient of x and y in parentheses:

9. $5ay + 3bx + 9cy - 3dx + x + y$.
10. $3abcx + 5cdy - 3x + 4y - x - y$.
11. $2x - 3ax - 2ay + 10by - cx + 3by + x$.

CHAPTER III.

MULTIPLICATION.

46. Multiplication is the process of taking one term as many times as there are units in another.

The quantity to be multiplied is called the **multiplicand**, and the quantity by which it is to be multiplied is called the **multiplier**.

The result is called the **product**.

Commutative Law.

$$10 \times 4 = 4 \times 10;$$
$$ab = ba.$$

Distributive Law.

$$10 \times 4 = (4 \times 5) + (4 \times 5);$$
$$ab + ac = a \times (b + c).$$

Associative Law.

$$10 \times (4 \times 5) = (10 \times 4) \times 5;$$
$$a \times (b \times c) = (a \times b) \times c.$$

Law of Coefficients.

47. Multiplying $+4$ by $+3$, we have,

$$4 + 4 + 4 = +12.$$

Again, -4 by $+3$ equals,

$$(-4) + (-4) + (-4) = -12.$$

Again, -4 by -3 equals,

$$-(-4) - (-4) - (-4) = +12.$$

Again, $+4$ by -3 equals,

$$-(+4) - (+4) - (+4) = -12.$$

Multiplying $5a$ by $3b$, we have,

$$3 \times 5 \times a \times b = 15ab.$$

Hence:

48. *The coefficient of the result is equal to the product of the coefficient of the multiplicand and the multiplier.*

Law of Exponents.

49. Multiplying a^3b^2 by a^5b^6, we have,
$a \times a \times a \times b \times b \times a \times a \times a \times a \times a \times b \times b \times b \times b \times b \times b$
$= aaabbaaaaabbbbbb = a^8b^8$.

Hence:

50. *The exponent of any letter in the result is equal to the sum of the exponents of this letter in both multiplier and multiplicand.*

Law of Signs.

51. By 47 we see that a plus quantity multiplied by a plus quantity gives a plus result; that a minus quantity multiplied by a minus quantity gives a plus result; that a plus quantity multiplied by a minus quantity gives a minus result; that a minus quantity multiplied by a plus quantity gives a minus result.

52. These results, tabulated in general form, give:

$+a \times (+b) = ab.$ $\qquad -a \times (+b) = -ab.$

$+a \times (-b) = -ab.$ $\qquad -a \times (-b) = +ab.$

Hence the rule:

53. I. *Like signs give a positive result.*

II. *Unlike signs give a negative result.*

54. The law holds true when applied to three or more factors, for the first two may be multiplied together to determine the sign, and this product, considered as a single factor, may be multiplied by the third factor, and so on.

55. Tabulating the results of three factors in general form, we have:

1. $+a \times +b \times +c$, all factors plus, $= +abc$.
2. $-a \times -b \times -c$, all factors minus, $= -abc$.
3. $+a \times +b \times -c$, two factors plus, $= -abc$.
4. $+a \times -b \times -c$, two factors minus, $= +abc$.

From the above we deduce the following rule:

56. I. *Positive factors give a positive result.*

II. *An even number of negative factors gives a positive result.*

III. *An odd number of negative factors gives a negative result.*

MONOMIALS BY MONOMIALS.

57. RULE.—*Find the product of the numerical coefficients, and to this product annex the different letters of each factor, giving to each letter an exponent equal to the sum of its exponents in the factors. Affix the proper sign.*

Examples.

1.	2.	3.	4.
$3a^2b$	$-3a^2b$	$-5abc^2$	$6x^2y^5z^7$
$5ab^2$	$7a^2b$	$-6ab^2c^3$	$-2xy^2z^8$
$15a^3b^3$	$-21a^4b^2$	$+30a^2b^3c^5$	$-12x^3y^7z^{15}$

EXERCISE 8.

5. Multiply $17x^2yz$ by $3x^3y^5z$.
6. Multiply $4abcd$ by $-6ab$.
7. Multiply $-81a^3b^6$ by $3ab^2$.
8. Multiply $-10m^2n$ by $-16mnp^2q$.
9. Multiply $-13x^2y^3z$ by $10x^3y^2z^{10}$.
10. Multiply $\frac{1}{2}abc^2$ by $\frac{3}{4}abc^3$.
11. Multiply a^x by b^y.
12. Multiply a^2b^m by $-a^0b^{mn}$.
13. Multiply $3a^{2+n}b^{m+3}$ by $-\frac{1}{3}a^{2-n}b^{m-3}$.

Find the continued product of the following:

14. $3a, 5abc, 6bc.$

15. $-5x^2, 6xy, 7x^2y^2z.$

16. $10c^2, -13c^3d^4, -14c^2d^2.$

17. $x^m, -y^n, -z^p.$

18. $-x^{2m+1}, y^{3m-1}, y^{-3m+1}.$

19. $\frac{1}{3}x^{0+2}, \frac{1}{2}x^{2m}, -z^{3p+q}.$

20. $3(a+b)^2, 5(a+b)^3, -6(a+b)^5.$

POLYNOMIALS BY MONOMIALS.

58. RULE.—*Multiply each term of the polynomial by the monomial, and add the partial products.*

Examples.

1.

$$\begin{array}{r} 4a+3b \\ 3a \\ \hline 12a^2+9ab \end{array}$$

2.

$$\begin{array}{r} 3x-4y^2 \\ -3z \\ \hline -9xz+12y^2z \end{array}$$

3.

$$\begin{array}{r} 6a+3b^3-4c^2 \\ 5d^2e \\ \hline 30ad^2e+15b^3d^2e-20c^2d^2e \end{array}$$

EXERCISE 9.

Multiply:

4. $9a+4b$ by $6abc.$

5. $6x^2-3y^4$ by $-3z^5.$

6. $3a^2b-6b^2c+7c^2d$ by $a^2b^2c^2.$

7. $\frac{1}{2}a-\frac{1}{3}b-\frac{1}{4}c$ by $ab.$

8. $\frac{2}{3}x+\frac{2}{5}y-\frac{3}{7}z$ by $\frac{1}{2}xy.$

9. $a^n+b^m-c^p$ by $-a^2b^2c^2.$

10. $a^{m+2}-b^{0+3}+c^{3+m}$ by $+a^nb^2c^3.$

POLYNOMIALS BY POLYNOMIALS.

59. RULE. — *Multiply all the terms of the multiplicand by each term of the multiplier, and add the partial products.*

1.
$$\begin{array}{r}
6x - 3y \\
4x^2 + 2y^3 \\
\hline
24x^3 - 12x^2y \qquad\qquad \\
+ 12xy^3 - 6y^4 \\
\hline
24x^3 - 12x^2y + 12xy^3 - 6y^4
\end{array}$$

2.
$$\begin{array}{r}
a^2b - b^3c + c^4d + abc \\
ab^2 + bc^3 - cd^4 \\
\hline
a^3b^3 - ab^5c + ab^2c^4d + a^2b^3c \\
a^2b^2c^3 - b^4c^4 + bc^7d + ab^2c^4 \\
- a^2bcd^4 + b^3c^2d^4 - c^5d^5 - abc^2d^4 \\
\hline
a^3b^3 - ab^5c + ab^2c^4d + a^2b^3c + a^2b^2c^3 - b^4c^4 + bc^7d \\
+ ab^2c^4 - a^2bcd^4 + b^3c^2d^4 - c^5d^5 - abc^2d^4.
\end{array}$$

EXERCISE 10.

Multiply:

3. $a^2 + 11$ by $a^2 - 9$.

5. $x^2 + 2xy + y^2$ by $x + y$.

4. $a^2 + a + 3$ by $a + 6$.

6. $3a + 4b - 3c$ by $5a + 4c$.

7. $a^2 + a + 1$ by $a^2 - a + 1$.

8. $-x^3 + x^2 - x + 1$ by $x + 1$.

9. $a^4 - a^3 + a^2 - a + 1$ by $a + 1$.

10. $3x^2 - 2y^3 + 3z^4$ by $x^2 - 6y^2 - 8z^5$.

11. $a^2 - ab + b^2 + a + b + 1$ by $a + b - 1$.

12. $\frac{1}{2}x^2 + \frac{1}{3}y^3 - \frac{1}{4}z^4$ by $\frac{1}{2}x - \frac{1}{3}y + \frac{1}{4}z$.

13. $a^mb^n + a^nb^m$ by $a^mb^n - a^nb^m$.

14. $x^{2p} - x^py^q + y^{2q}$ by $x^{2p} + x^py^q + y^{2q}$.

Find the product of:

15. $a + x,\ a + y,\ a + z.$

16. $a^2 + a + 1,\ a^2 - a + 1,\ a^4 - a^2 + 1.$

17. $a^2 + a + 1,\ a - 1,\ a^3 - 1.$

18. $a - x,\ a + x,\ a^2 + x^2,\ a^4 + x^4.$

60. Sometimes it is convenient simply to indicate the multiplication of two quantities. This is done by enclosing the quantities in parentheses, writing one after the other.

When the multiplication indicated is actually performed, the expression is said to be simplified.

EXERCISE 11.

Perform the operation indicated:

1. $(a + b)(a + b).$

3. $(a + b)(a - b).$

2. $(a - b)(a - b).$

4. $(a^2 - a + 1)(a + 1).$

5. $(a^2 + a + 1)(a - 1).$

6. $(a^2 + a + 1)(a^2 - a + 1).$

7. $(a + b)(a + b) - (a - b)(a - b) + (a + b)(a - b).$

8. $(x + \frac{2}{3})(x - \frac{2}{3})(x^2 + \frac{4}{9}).$

9. $(x^{2m} + x^m y^n + y^{2n})(x^{2m} - x^m y^n + y^{2n})(x^{4m} - x^{2m} y^{2n} + y^{4n}).$

10. $(a + b + 1)(a + b - 1)(a^2 + b^2 + 2b + 1).$

61. The three following theorems are useful in the multiplication of binomials.

I. $(a + b)^2 = a^2 + 2ab + b^2.$

That is:

62. *The square of the sum of two quantities is equal to the square of the first, plus twice the product of the first by the second, plus the square of the second.*

II. $(a-b)^2 = a^2 - 2ab + b^2$.

That is:

63. *The square of the difference of two quantities is equal to the square of the first, minus twice the product of the first by the second, plus the square of the second.*

III. $(a+b)(a-b) = a^2 - b^2$.

That is:

64. *The product of the sum and difference of two quantities is equal to the difference of their squares.*

Applying the principles involved in these three theorems, perform the operations indicated in the following examples:

EXERCISE 12.

1. $(a+2b)^2$.
2. $(a-4b)^2$.
3. $(x^2+5y)^2$.
4. $(x^2-11y)^2$.
5. $(3a^2+4b^2)^2$.
6. $(9a^3-5b^3)^2$.
7. $(\frac{1}{2}a+\frac{1}{3}b)^2$.
8. $(\frac{1}{2}x^2-\frac{1}{4}y)^2$.
9. $(a^{\frac{1}{2}}+b^{\frac{1}{2}})^2$.
10. $(3x^{\frac{1}{3}}-2y^{\frac{1}{3}})^2$.
11. $(x^a+y^b)^2$.
12. $(a^{-x}-b^{-y})^2$.
13. $(2a+3)(2a-3)$.
14. $(2x+5y)(2x-5y)$.
15. $(x^2\times y^2)(x^2-y^2)^2$.
16. $(a+\frac{1}{2})(a-\frac{1}{2})$.
17. $(\frac{1}{2}a-1)(\frac{1}{2}a+1)$.
18. $(a^m+1)(a^m-1)$.
19. $(\frac{1}{3}+x^{\frac{1}{3}})(\frac{1}{3}-x^{\frac{1}{3}})$.
20. $(1+\frac{1}{2}a^{\frac{1}{2}})(1-\frac{1}{2}a^{\frac{1}{2}})$.
21. $[(a+b)+(a-b)][(a+b)-(a-b)]$.
22. $[(x+1)+(y+1)][(x+1)-(y+1)]$.
23. $[(3+a)+(2-b)][(3+a)-(2-b)]$.
24. $[(a-x)^2+(a-y)^2][(a-x)^2-(a-y)^2]$.

CHAPTER IV.

DIVISION.

65. Division is the process of finding how many times one quantity called the **divisor** is contained in another called the **dividend.** The result is called the **quotient.**

It follows therefore that division is the reverse of multiplication.

66. Law of Signs.

Since by multiplication

$$\begin{array}{rclcrcl} a \times b &=& ab, & \text{therefore} & ab \div a &=& b; \\ a \times -b &=& -ab, & \text{“} & -ab \div a &=& -b; \\ -a \times +b &=& -ab, & \text{“} & -ab \div -a &=& +b; \\ -a \times -b &=& +ab, & \text{“} & +ab \div -a &=& -b. \end{array}$$

67. That is:

I. *Like signs in dividend and divisor give plus in the quotient.*

II. *And unlike signs in dividend and divisor give minus in the quotient.*

68. Law of Coefficients.

In multiplication it was shown that the coefficient of the product is equal to the coefficient of the multiplicand multiplied by that of the multiplier; hence it follows in division:

That the coefficient of the quotient is equal to the coefficient of the dividend divided by that of the divisor.

69. Law of Exponents.

In multiplication it was shown that the exponent of the product is equal to the exponent of the multiplicand plus the exponent of the multiplier; hence it follows in division:

That the exponent of the quotient is equal to the exponent of the dividend minus that of the divisor.

Thus, $4a^2 \times 2a = 8a^3;$

$\therefore\ 8a^3 \div 2a = 4a^2.$

Again, $12\,abcd \times 3\,xy = 36\,abcdxy;$

$\therefore\ 36\,abcdxy \div 3\,xy = 12\,abcd.$

Division of One Monomial by Another.

70. Rule. — *Write the dividend over the divisor. Cancel the common factors.*

Examples.

1. $\dfrac{12a^2b}{3ab} = 4a.$

2. $\dfrac{16x^3y^5}{8x^2y^2} = 2xy^3.$

3. $\dfrac{2a^{2m}b^{3n}}{a^mb^{2n}} = 2a^mb^n.$

4. $\dfrac{3x^{a+3}}{x^a} = 3x^3.$

5. $\dfrac{20a^{-1}b^{-5}}{4a^{-1}b^{-3}} = 5b^{-2}.$

6. $\dfrac{6a^{x+y}b^{x-y}}{3a^{x+y}b^{x-y}} = 2.$

EXERCISE 13.

Divide:

7. $25\,abcd$ by $5\,acd.$
8. $32a^2b^2c^5$ by $8c^5.$
9. $-10x^2y$ by $-2x^2y.$
10. $-20x^3y^2$ by $x^2y.$
11. $-a^7b^6$ by $a^5b^5.$
12. $a^{10}c^4$ by $-a^2c.$
13. $3x^my^n$ by $3x^m.$
14. $28a^{n-2}b^{m-3}$ by $7a^{n-2}.$
15. a^p by $a^2.$
16. a^{-p} by $a^{p+3}.$
17. $5x^{2-m}y^{n-2}$ by $x^{m+3}y^{3-n}.$
18. $-10a^3bx^{3+p}$ by $5a^2x^{3+p}.$
19. $a^{\frac{3}{4}}b^{\frac{3}{2}}c^{\frac{5}{4}}$ by $a^{\frac{1}{2}}b^{\frac{1}{4}}c^{\frac{1}{8}}.$

ZERO POWERS AND RECIPROCALS.

71. I. To prove $a^0 = 1$.

For $\frac{a}{a} = 1$; but $\frac{a}{a} = a^{1-1} = a^0 = 1$.

72. II. To prove $a^{-1} = \frac{1}{a}$.

For $a^{-1} = \frac{a^{-1}}{1}$. Multiplying both terms of this fraction by a,

we have, $$\frac{a^{-1}}{1} \times \frac{a}{a} = \frac{a^{-1+1}}{a} = \frac{a^0}{a} = \frac{1}{a}.$$

73. From these two theorems we deduce the following rules:

I. *Any quantity which has zero for its exponent is equal to unity.*

II. *Any quantity with a negative exponent is equal to the reciprocal of that quantity with an equal positive exponent.*

DIVISION OF A POLYNOMIAL BY A MONOMIAL.

74. RULE. — *Divide each term of the polynomial by the monomial, and connect the partial quotients with the proper sign.*

Examples.

1. $$\frac{3abc + 30ab^2x - 24ac^2x}{3a} = \frac{3abc}{3a} + \frac{30ab^2x}{3a} - \frac{24ac^2x}{3a}$$
$$= bc + 10b^2x - 8c^2x.$$

EXERCISE 14.

Divide:

2. $25ax^2y^4 - 30x^5y^6 + 45x^7y^{10}$ by $5x^2y^4$.
3. $\frac{3}{4}a^5b^7 + \frac{5}{8}a^{10}b^{12} - \frac{3}{16}a^8b^9$ by $\frac{1}{4}a^4b^4$.
4. $ab^{2n+1} + a^2b^{2n} + a^3b^{2n-1}$ by ab^{n+1}.
5. $3m^{x+1}n^{y+1} - 4m^xn^{y+2}$ by $6m^2n^y$.
6. $a^2 + a + 1$ by a^{-2}.
7. $x^{-6}y^2 + x^{-4}y^4 + x^{-2}y^6$ by $x^{-2}y^2$.

DIVISION OF ONE POLYNOMIAL BY ANOTHER.

Divide $12a^8 + 2a^6b^2 - 40a^4b^4 + 34a^2b^6 - 8b^8$

by $4a^4 - 6a^2b^2 + 2b^4$.

(1)
$$\begin{array}{l|l}
12a^8 + 2a^6b^2 - 40a^4b^4 + 34a^2b^6 - 8b^8 & 4a^4 - 6a^2b^2 + 2b^4 \\ \cline{2-2}
\underline{12a^8 - 18a^6b^2 + 6a^4b^4} & 3a^4 + 5a^2b^2 - 4b^4 \\
\qquad 20a^6b^2 - 46a^4b^4 + 34a^2b^6 & \\
\qquad \underline{20a^6b^2 - 30a^4b^4 + 10a^2b^6} & \\
\qquad\qquad -16a^4b^4 + 24a^2b^6 - 8b^8 & \\
\qquad\qquad \underline{-16a^4b^4 + 24a^2b^6 - 8b^8} &
\end{array}$$

(2)
$$\begin{array}{l|l}
1 + x^2 + x^4 & 1 - x + x^2 \\ \cline{2-2}
\underline{1 - x + x^2} & 1 + x + x^2 \\
x + x^4 & \\
\underline{x - x^2 + x^3} & \\
\quad x^2 - x^3 + x^4 & \\
\quad \underline{x^2 - x^3 + x^4} &
\end{array}$$

(3)
$$\begin{array}{l|l}
\frac{16}{3}x^4 - \frac{22}{3}x^3 + \frac{41}{2}x^2 - \frac{23}{2}x + 6 & 2x^2 - \frac{3}{2}x + 6 \\ \cline{2-2}
\underline{\frac{16}{3}x^4 - \frac{24}{6}x^3 + \frac{48}{3}x^2} & \frac{8}{3}x^2 - \frac{5}{3}x + 1 \\
\qquad -\frac{10}{3}x^3 + \frac{9}{2}x^2 - \frac{23}{2}x & \\
\qquad \underline{-\frac{10}{3}x^3 + \frac{15}{6}x^2 - \frac{30}{3}x} & \\
\qquad\qquad + 2x^2 - \frac{3}{2}x + 6 & \\
\qquad\qquad \underline{+ 2x^2 - \frac{3}{2}x + 6} &
\end{array}$$

75. From these examples we deduce the following rule:

I. *Arrange both dividend and divisor according to the descending or ascending powers of one of the letters.*

II. *Divide the first term of the dividend by the first term of the divisor, and write the result as the first term of the quotient.*

III. *Multiply the whole divisor by the term of the quotient thus found, and subtract this product from the dividend.*

IV. *Arrange the remainder as a new dividend and proceed as before.*

76. The student will readily observe that it greatly facilitates the process to arrange the dividend and divisor in the same order as regards some common letter, and in performing the operation of subtraction for the partial dividends that this order should be maintained throughout.

EXERCISE 15.

Divide:

1. $x^2 + 12x + 32$ by $x + 4$.
2. $x^2 - 16x + 60$ by $x - 6$.
3. $x^3 - 3x^2 + 3x - 1$ by $x^2 - 2x + 1$.
4. $x^4 + x^2 + 1$ by $x^2 - x + 1$.
5. $2ab^2 + b^3 + a^3 + 2a^2b$ by $a^2 + b^2 + ab$.
6. $x^6 + y^6 + 2x^3y^3$ by $x^2 + y^2 - xy$.
7. $a^6 + b^6 + c^6 - 3a^2b^2c^2$ by $a^2 + b^2 + c^2$.
8. $6a^4 - a^3b + 4b^4 + 2a^2b^2 + 13ab^3$ by $2a^2 - 3ab + 4b^2$.
9. $x^3 - 4x^2 + 5x - 2$ by $x - 1$.
10. $a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$ by $a^2 + 2ab + b^2$.
11. $x^3 + y^3 + z^3 - 3xyz$ by $x^2 + y^2 + z^2 - xy - xz - yz$.
12. $a^3 - b^3 + c^3 + 3b^2c - 3bc^2$ by $a - b + c$.
13. $a^{6n} + b^{6n}$ by $a^n + b^n$.
14. $x^{4a+4} + x^{2a+2}y^{2b+2} + y^{4b+4}$ by $x^{2a+2} + x^{a+1}y^{b+1} + y^{2b+2}$.
15. $a^{2n+1} - a^{2n-1} + a^{n+2} + a^n - a^2$ by $a^{n+1} + a^n - a$.

CHAPTER V.

SIMPLE EQUATIONS.

77. Equations. An equation is an expression of equality between two quantities.

Thus, $a + b = 3$

is an equation signifying that the sum of a and b is equal to 3.

78. The quantity on the left of the sign of equality is called the **first member** of an equation, and the quantity on the right of the sign of equality is called the **second member** of an equation.

79. Equations are employed to find the value of an unknown quantity.

80. Unknown Quantities are generally represented by the last letters of the alphabet, as x, y, z, etc.

81. Known Quantities are represented by the first letters of the alphabet, as a, b, c, d, etc.

Thus in the equation

$$abx + cdx = 5e,$$

x represents the unknown quantity, and a, b, c, d, and e the known quantities.

82. A **Numerical Equation** is one in which all of the known quantities are expressed by figures.

Thus, $3x + 4x = 28.$

83. A **Literal Equation** is one in which all of the known quantities are expressed by letters.

Thus, $$abx + cdx = abc.$$

84. An **Identical Equation** is one in which the two members are the same, or reducible to the same algebraic expression.

Thus, $$3x + 4y^2 = 3x + 4y^2.$$

85. An **Equation of Condition** is one which is true only for certain values of the letters involved.

Thus, and $$\left.\begin{aligned} 3x - 5 &= 10 \\ x + 5 &= 10 \end{aligned}\right\} \text{are true when } x = 5.$$

86. The **Degree** of an equation is determined by the highest sum of the exponents of the unknown quantities in any term.

87. A **Simple Equation** is one of the first degree.

Thus, and $$\left.\begin{aligned} 3x + b &= 21 \\ 3ax + c &= b \end{aligned}\right\} \text{are equations of the first degree.}$$

88. **Transformation** of an equation is the process of changing the form without destroying the equality so as to determine the value of the unknown quantities.

89. In solving equations, the following truths are useful:

I. *Equal quantities may be added to both members.*

II. *Equal quantities may be subtracted from both members.*

III. *Both members may be multiplied or divided by equal quantities.*

90. **Transposition of Terms.** In solving an equation all the terms containing the unknown quantity should be brought to the first member, and all the other terms to the second member.

Thus, $$(1)\ 3x - 7 = -4x + 14.$$

Adding $4x$ to both sides, we have,

$$(2)\ 7x - 7 = 14.$$

Again adding 7 to both sides, we have,

$$(3)\ 7x = 21.$$

Dividing both sides by 7, we have,

$$(4)\ x = 3.$$

Substituting this value of x in equation (1), we have the identity

$$(5)\ 2 = 2.$$

By comparing the several results obtained in the solving of the given equation, we may deduce the following rules:

91. I. *Any term may be transposed from one member of an equation to the other if its sign be changed.*

II. *The sign of every term of the equation may be changed, for this is equivalent to multiplying both members by* -1.

EXERCISE 16.

1. $10x - 2 = 38.$
2. $48x = 14x + 68.$
3. $24 - 10x = 38 - 24x.$
4. $10x - 2(3x - 7) = 8x - 2(6x - 35).$
5. $4(x - 3)(x + 7) = 4(x - 15)(x - 5).$
6. $5a^2x + 5bc = 5ab - 10ax.$
7. $3x - 3ab - 6cx = 9b - 3a + 12c^2x.$
8. $6ax - 6c^2x = -6bc + 6a^2.$
9. $16(3 - x) = 8x + - 12 + 6x.$
10. $2x(x - 2)(x - 1) = 2x(x - 7)(x - 8).$
11. $x + 1 + (x^2 - 1) = (x - 2)(x + 5).$
12. $(12x - a) + 6 = 13x - (2a - 6).$

PROBLEMS.

Solution of Problems.

(1) A is twice as old as B; the sum of their ages is 48 years. How old is each?

Let x equal the number of years of B's age.

Then $2x$ will equal number of years of A's age,

and $x + 2x$ will equal 48.

Hence the equation

$$x + 2x = 48.$$

Combining, $3x = 48.$

Dividing by 3 $x = 16.$

Multiplying by 2 $2x = 32.$

$\therefore$ A is 32 years old, and B is 16 years old.

EXERCISE 17.

1. If I add 16 to a certain number, my answer is 21. What is the number?

2. To four times a certain number I add 36, and obtain 76 for my answer. What is the number?

3. The sum of the ages of John and James is 48 years. John is twice as old as James. How old is each?

4. In a certain school there are 350 pupils. If 50 be subtracted from the number of boys, it will still be twice as great as the number of girls. How many boys and how many girls in the school?

5. A school gives $150 for four prizes. The first is double the second; the second is double the third; the third is double the fourth. What is the amount of each prize?

6. One number is eight times another; if I subtract -5 from the first, and add 40 to the second, the results will be equal. What are the numbers?

7. Thirty cows, 40 sheep, and 50 horses cost $3450. A cow costs five times as much as a sheep, and a horse costs twice as much as a cow. What is the price of each?

8. If I add 40 to a certain number, the sum will be as much above 70 as the number is below 70. What is the number?

9. In a theatre there are 700 people: men, women, and children. There are one half as many women as men, and four times as many children as women. How many are there of each?

10. A lady is three times as old as her daughter; five years ago she was five times as old. How old is each?

11. The joint age of a father and two sons, one of whom is two years older than the other, is 62 years; if 26 years be added to the father's age, it will equal three times the combined ages of the sons.

12. Find two numbers differing by 25 so that six times the less will equal three times the greater.

13. John has 75 dollars, and James has 225 dollars. John gives James a certain sum, and then James has five times as much money as John. How much did John give to James?

14. A cistern holding 300 gallons is filled with two pipes, the second carrying five times the volume of the first. How many gallons does each pipe put into the cistern?

15. Divide 125 into two parts so that six times one shall be equal to nine times the other.

16. A sum of money amounting to $60 consists of twenty-five-cent pieces, fifty-cent pieces, and one-dollar pieces. There are twice as many fifty-cent pieces as twenty-five-cent pieces, and in all there are 88 coins. How many are there of each?

CHAPTER VI.

FACTORS.

A thorough knowledge of factors will greatly aid the student in performing complicated operations in Algebra with a degree of rapidity.

The following chapter will be devoted to factors most useful for future work.

92. Factors of a quantity are those quantities which, multiplied together, produce the given quantity.

Thus, 4 and 4 are factors of 16.

Again, $2a$ and $6a$ are factors of $12a^2$.

Again, ab and $\sqrt{2}$ are factors of $ab\sqrt{2}$.

93. A Prime Factor is a quantity which is divisible only by itself and unity.

Thus, 3, 2, and 2 are the prime factors of 12.

Again, a, b, and c are the prime factors of abc.

Again, a, b, and $\sqrt{2}$ are the prime factors of $ab\sqrt{2}$.

94. I. **Monomials.** Monomials can, in most cases, be factored by inspection.

Thus, $8a^2b^3c = 2 \times 2 \times 2 \times a \times a \times b \times b \times b \times c$.

In writing the factors of a monomial expression we generally take the prime factors of the numerical part and the several letters with their exponents attached.

Thus, $16x^4y^6 = 2 \times 2 \times 2 \times 2 \times x^4 \times y^6$.

Again,

$144a^5b^2\sqrt{a+b} = 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times a^5 \times b^2 \times \sqrt{a+b}$.

A rational factor contains no radical part.

EXERCISE 18.

Factor:

1. $4a^2b^2$.
2. $16a^4b^4$.
3. $125x^3y^3$.
4. $256x^4y^4$.
5. $100a^{2n}b^{2m}$.
6. $10x^{4a}y^{4b}$.
7. $64x^{\frac{2}{3}}y^{\frac{4}{5}}z^{\frac{4}{5}}$.
8. $100a^{2+2m}b^{2+2n}$.
9. $50x^{3a+6b}y^{6a+3b}$.
10. $20a^{-4+2n}b^{-4+2m}$.

95. II. **When each term of a polynomial has a common factor.**

Thus, $3ax^2y^3 - 9axy^4 = 3axy^3(x - 3y)$.

EXERCISE 19.

Factor:

1. $16a^2 - 4a$.
2. $9x^3 - 6x^2 - 3x$.
3. $8ax^2 + 4a^2x - 2ax$.
4. $25abcd + 20abc + 15ab$.
5. $125x^4y^8z^6 + 250xy - 500yz$.
6. $a^{2n} + a^nb + a^{4n}bc$.
7. $x^{4a}y^{2b} - x^{2a}y^b + x^ay^0$.
8. $a^{\frac{2}{3}} + a^{\frac{1}{3}} - ab + a^2b^2$.
9. $4x^{2n+4} - 8x^{4n+8}yz$.
10. $100x^{-2}y^{-4} + x^{-4}y^{-5} - 200x^{-8}y^{-6}$.

96. III. **When the terms of a polynomial can be grouped so as to give a common binomial factor.**

Thus,

$$\begin{aligned} &ax + ay + bx + by \\ &= a(x + y) + b(x + y) \\ &= (a + b)(x + y). \end{aligned}$$

EXERCISE 20.

Factor:

1. $2x + 2y + ax + ay$.
2. $x^2 - ax - bx + ab$.
3. $mn - ma - na + a^2$.
4. $abcd + abcx + d^2 + xd$.
5. $a^2c + ad + a^3 + cd$.
6. $x^ab^x + x^ac^y + y^bb^x + y^bc^y$.
7. $x^{2m} - ax^m - x^my^n + ay^n$.
8. $a^{-2}x^{-1} + a^{-2}y^{-1} - b^{-2}y^{-1} - b^{-2}x^{-1}$.
9. $\frac{1}{4}ac - \frac{1}{6}bc - \frac{1}{6}ad + \frac{1}{9}bd$.
10. $x^{-4} - a^ax^{-2} - b^bx^{-2} + a^ab^b$.

97. IV. **When the terms of a polynomial can be grouped so as to give a common trinomial factor.**

Thus,

$$\begin{aligned}&ax + ay + az - x^2 - xy - xz\\ &= a(x + y + z) - x(x + y + z)\\ &= (a - x)(x + y + z).\end{aligned}$$

EXERCISE 21.

Factor:

1. $a^2c^2 + bc^2 + a^2d + bd + a^2e + be$.
2. $x^3 - xy^2 + x^2y - y^3 + x^2z - y^2z$.
3. $mp^2 - np^2 - mq + nq - nr^3 + mr^3$.
4. $a^2x + a^2y + a^2z - b^3x - b^3y - b^3z + c^4x + c^4y + c^4z$.
5. $a^2bx - abcx + ab^2y - b^2cy - abcz + bc^2z$.
6. $a^2bp^2q^2 - a^2bm^2n^2 - b^2cp^2q^2 + b^2cm^2n^2 + c^2dm^2n^2 - c^2dp^2q^2$.
7. $a^nx^a + b^mx^a + a^ny^b + b^my^b + a^nz^c + b^mz^c$.
8. $12\,ax^2 + 15\,bx^2 + 16\,ay^3 + 20\,by^3 + 20\,az^4 + 25\,bz^4$.
9. $\frac{1}{4}px - \frac{1}{6}py + \frac{1}{6}qx - \frac{1}{9}qy + \frac{1}{8}rx - \frac{1}{12}ry$.
10. $a^{-2}x^{-5} - b^{-3}x^{-5} - c^{-4}x^{-5} - a^{-2}y^{-6} + b^{-3}y^{-6} + c^{-4}y^{-6}$.

98. V. **When a trinomial is a perfect square in the form of $a^2 + 2ab + b^2$.**

By (62), $(a + b)(a + b) = a^2 + 2ab + b^2$.

Conversely, $a^2 + 2ab + b^2 = (a + b)(a + b)$.

99. Hence, *a trinomial in the form of $a^2 + 2ab + b^2$ can be resolved into two binomial factors each of which is the sum of the square roots of the first and last terms.*

EXERCISE 22.

Factor:

1. $a^2 + 6a + 9$.
2. $x^2 + 24x + 144$.
3. $a^2 + 40a + 400$.
4. $4x^2 + 12xy + 9y^2$.

5. $x^2 + 2x + 1$.
6. $a^{2m} + 2a^m b^n + b^{2n}$.
7. $\frac{1}{4}a^2 + \frac{1}{2}ab + \frac{1}{4}b^2$.
8. $16a^2x^2 + 72axy + 81y^2$.
9. $x^{-4} + 2x^{-2}y^{-2} + y^{\ 4}$.
10. $9a^{-4x} + 54a^{-2x}b^{-2} + 81b^{-4}$.

100. VI. **When a trinomial is a perfect square in the form of $a^2 - 2ab + b^2$.**

By (63), $(a - b)(a - b) = a^2 - 2ab + b^2$.

Conversely, $a^2 - 2ab + b^2 = (a - b)(a - b)$.

101. Hence, *a trinomial in the form of* $a^2 - 2ab + b^2$ *can be resolved into two binomial factors each of which is the difference of the square roots of the first and last terms.*

EXERCISE 23.

Factor:

1. $a^2 - 12a + 36$.
2. $x^2 - 34x + 289$.
3. $9x^2 - 36xy + 36y^2$.
4. $256x^4 - 32x^2 + 1$.
5. $4a^4x^4 - 8a^2b^2x^2y^2 + 4b^4y^4$.
6. $x^{2a} - 2x^a y^b + y^{2b}$.
7. $\frac{1}{9}x^4 - \frac{1}{6}x^2y^2 + \frac{1}{16}y^4$.
8. $a^{2m} - a^m + 1$.
9. $x^{-4} - 2x^{-2}y^{-2} + y^{-4}$.
10. $x^{-2a} - 2x^{-a}y^{-b} + y^{-2b}$.

102. VII. **When a binomial is in the form of $a^2 - b^2$.**

By (64), $(a + b)(a - b) = a^2 - b^2$.

Conversely, $a^2 - b^2 = (a + b)(a - b)$.

103. Hence, *a binomial in the form of* $a^2 - b^2$ *can be resolved into two binomial factors, one of which is the sum of the square roots of the first and second terms, and the other the difference of these roots.*

EXERCISE 24.

Factor:

1. $x^2 - 4y^2$.
2. $36x^2 - 1$.
3. $25a^2 - 81b^2$.
4. $1 - a^2b^2$.
5. $x^2y^2 - 9$.
6. $\frac{1}{4}m^2 - n^2$.
7. $m^{-2} - n^{-2}$.
8. $x^{-2a} - y^{-2a}$.
9. $a^{2x-2} - b^2$.
10. $x^{2(m+n)} - y^2$.

104. VIII. **When a quantity can be reduced to the form of $a^2 - b^2$.**

Thus, $a^8 - b^8 = [(a^4)^2 - (b^4)^2]$.

Factor:

(1) $a^8 - b^8 = (a^4 + b^4)(a^4 - b^4)$;

but $a^4 - b^4 = (a^2 + b^2)(a^2 - b^2)$;

but $a^2 - b^2 = (a + b)(a - b)$.

Hence the factors of $a^8 - b^8$

$$= (a^4 + b^4)(a^2 + b^2)(a + b)(a - b).$$

(2) $a^2 + 2ab + b^2 - c^2$.

$$\begin{aligned} a^2 + 2ab + b^2 - c^2 &= (a^2 + 2ab + b^2) - c^2 \\ &= (a + b)^2 - c^2 \\ &= \{(a + b) + c\}\{(a + b) - c\} \\ &= (a + b + c)(a + b - c). \end{aligned}$$

(3) $1 - 2xy - x^2 - y^2$.

$$\begin{aligned} 1 - 2xy - x^2 - y^2 &= 1 - (x^2 + 2xy + y^2) \\ &= 1 - (x + y)^2 \\ &= \{1 + (x + y)\}\{1 - (x + y)\} \\ &= (1 + x + y)(1 - x - y). \end{aligned}$$

(4) $a^2 - 2ab + b^2 - x^2 - 2xy - y^2$.

$$\begin{aligned} a^2+2ab+b^2-x^2-2xy-y^2 &= (a^2+2ab+b^2)-(x^2+2xy+y^2) \\ &= (a+b)^2-(x+y)^2 \\ &= \{(a+b)+(x+y)\}\{(a+b)-(x+y)\} \\ &= (a+b+x+y)(a+b-x-y). \end{aligned}$$

(5) $81a^2 - 9x^2 - 24xy - 16y^2$.

$$\begin{aligned} 81a^2-9x^2-24xy-16y^2 &= 81a^2-(9x^2+24xy+16y^2) \\ &= \{9a+(3x+4y)\}\{9a-(3x+4y)\} \\ &= (9a+3x+4y)(9a-3x-4y). \end{aligned}$$

EXERCISE 25.

Factor:

1. $a^{16} - 1$.
2. $1 - x^{32}$.
3. $256a^4 - b^4$.
4. $81x^4 - 625y^4$.
5. $a^{4x} - b^{4y}$.
6. $a^{-8} - 1$.
7. $1 - a^2 - 2ab - b^2$.
8. $(a-1)^4 - (b-1)^4$.
9. $4x^2 + 4x + 1 - z^2$.
10. $(x^2+y^2+z^2)^2 - (a^2+b^2+c^2)^2$.
11. $a^2 + b^2 - x^2 - y^2 + 2ab - 2xy$.
12. $4m^2 - 8mn + 4n^2 - 16p^2 + 32pq - 16q^2$.
13. $a^{2m} - 2a^mb^n + b^{2n} - x^{2a} - 2x^ay^b - y^{2b}$.
14. $x^{-2} - 2x^{-1}y^{-1} + y^{-2} - a^{-2} - 2a^{-1} - 1$.
15. $m^{-2a} - 2m^{-a} + 1 - p^{-4a} + 2p^{-2a} - 1$.

105. IX. **When a trinomial is in the form of $x^4 + x^2y^2 + y^4$.**

Factor: (1) $a^4 + a^2b^2 + b^4$.

By adding and subtracting a^2b^2, we have $a^4 + 2a^2b^2 + b^4 - a^2b^2$, the value of which is equal to $a^4 + a^2b^2 + b^4$.

$$\begin{aligned}\text{Hence, } a^4+a^2b^2+b^4 &= a^4+2a^2b^2+b^4-a^2b^2\\ &=(a^2+b^2)^2-a^2b^2\\ &=\{(a^2+b^2)+ab\}\{(a^2+b^2)-ab\}\\ &=(a^2+ab+b^2)(a^2-ab+b^2).\end{aligned}$$

$$\begin{aligned}(2)\qquad 16a^4+4a^2+1 &= 16a^4+8a^2+1-4a^2\\ &=(4a^2+1)^2-4a^2\\ &=\{(4a^2+1)+2a\}\{(4a^2+1)-2a\}\\ &=(4a^2+2a+1)(4a^2-2a+1).\end{aligned}$$

Certain trinomials, though not strictly of the form of $x^4+x^2y^2+y^4$, may be factored by the same process; that is, by adding and subtracting a perfect square, such that the resulting quantity will be in the form of the difference between two squares.

Factor: (1) $9x^4+41x^2y^2+49y^4$.

Adding and subtracting x^2y^2, we have,

$$9x^4+42x^2y^2+49y^4-x^2y^2,$$

which equals

$$\begin{aligned}&(9x^4+42x^2y^2+49y^4)-x^2y^2\\ &\quad=(3x^2+7y^2)^2-x^2y^2\\ &\quad=\{(3x^2+7y^2)+xy\}\{(3x^2+7y^2)-xy\}\\ &\quad=(3x^2+xy+7y^2)(3x^2-xy+7y^2).\end{aligned}$$

106. Some trinomials of this form admit of two solutions.

Thus, $16x^4-17x^2y^2+y^4$ may be factored

into $(4x^2+3xy-y^2)(4x^2-3xy-y^2)$,

and $(4x^2+5xy+y^2)(4x^2-5xy+y^2)$.

Factor:

$$\begin{aligned}(a)\ 16x^4-17x^2y^2+y^4 &= 16x^4-8x^2y^2+y^4-9x^2y^2\\ &=(4x^2-y^2)^2-9x^2y^2\\ &=(4x^2-y^2-3xy)(4x^2-y^2+3xy)\\ &=(4x^2-3xy-y^2)(4x^2+3xy-y^2).\end{aligned}$$

(b) $16x^4 - 17x^2y^2 + y^4 = 16x^4 + 8x^2y^2 + y^4 - 25x^2y^2$

$$= (4x^2 + y^2)^2 - 25x^2y^2$$
$$= (4x^2 + y^2 + 5xy)(4x^2 + y^2 - 5xy)$$
$$= (4x^2 + 5xy + y^2)(4x^2 - 5xy + y^2).$$

EXERCISE 26.

Factor. One solution.

1. $a^4 + a^2b^2 + b^4$.
2. $16x^4 + 7x^2y^2 + y^4$.
3. $25a^4b^4 + 19a^2b^2c^2d^2 + 4c^4d^4$.
4. $36x^4 + 23x^2y^2 + 4y^4$.
5. $49a^4 + 33a^2b^2 + 9b^4$.
6. $81x^4 - 28x^2y^2 + 16y^4$.
7. $144a^{4x} - a^{2x}b^{2y} + b^{4y}$.
8. $169x^{-4} - 48x^{\ 2}y^{-2} + 4y^{-4}$.
9. $64a^{4m} + 7a^{2m}b^{2n} + 64b^{4n}$.
10. $9x^{-4} + 41x^{-2}y^{-2} + 49y^{-4}$.

EXERCISE 27.

Factor. Two solutions.

1. $4x^4 - 5x^2y^2 + y^4$.
2. $16a^4 - 17a^2 + 1$.
3. $9x^4 - 10x^2y^2 + y^4$.
4. $25a^4b^4 - 26a^2b^2 + 1$.
5. $36x^4 - 37x^2y^2 + y^4$.
6. $9a^4 - 13a^2b^2 + 4b^4$.
7. $49x^4 - 65x^2y^2 + 16y^4$.
8. $64x^{4a} - 185x^{2a}y^{2b} + 121y^{4b}$.
9. $25a^{-4} - 74a^{\ 2}b^{\ 2} + 49b^{-4}$.
10. $81a^{-4x} - 130a^{\ 2x}b^{\ 2y} + 49b^{-4y}$.

NOTE. — Each of the above trinomials may be factored into binomials, but this process will be treated in Case XV.

107. X. **When a trinomial is in the form of $a^4 - a^2b^2 + b^4$.**

Factor:

(1) $a^4 - a^2b^2 + b^2 = a^4 + 2a^2b^2 + b^4 - 3a^2b^2$

$$= (a^4 + 2a^2b^2 + b^4) - 3a^2b^2$$
$$= (a^2 + b^2)^2 - 3a^2b^2$$
$$= (a^2 + b^2 + ab\sqrt{3})(a^2 + b^2 - ab\sqrt{3})$$
$$= (a^2 + ab\sqrt{3} + b^2)(a^2 - ab\sqrt{3} + b^2).$$

$$(2)\quad 4a^4 - 2a^2 + 1 = 4a^4 + 4a^2 + 1 - 6a^2$$
$$= (4a^4 + 4a^2 + 1) - 6a^2$$
$$= (2a^2 + 1)^2 - 6a^2$$
$$= (2a^2 + 1 + a\sqrt{6})(2a^2 + 1 - a\sqrt{6})$$
$$= (2a^2 + a\sqrt{6} + 1)(2a^2 - a\sqrt{6} + 1).$$

EXERCISE 28.

Factor:

1. $x^4 - x^2y^2 + y^4$.
2. $x^4 - 4x^2y^2 + y^4$.
3. $9x^4 - 12x^2y^2 + 16y^4$.
4. $9x^4 - 3x^2y^2 + y^4$.
5. $81a^4 - 46a^2b^2 + 9b^4$.
6. $256a^4 - 48a^2b^2 + 9b^4$.
7. $225x^4 - 106x^2y^2 + 25y^4$.
8. $a^{4x} - a^{2x} + 1$.
9. $81x^{-4} - 46x^{-2} + 9y^{-4}$.
10. $324a^{-4x} - 108x^{-2x} + 1$.

108. XI. **When a trinomial is in the form of $x^2 + ax + b$.**

Multiplying $(x + m)$ by $(x + n)$,

we have, $x^2 + (m + n)x + mn$.

Now if we put a for $m + n$, and b for mn,

we have, $x^2 + ax + b$.

109. Hence, *a trinomial of the form of $x^2 + ax + b$ can be resolved into two binomial factors:*

The first term of each of which is x.
The sum of the second term of which is a.
The product of the second term of which is b.

In this formula a and b stand for the sum and product respectively of any terms whatsoever.

Factor: (1) $x^2 + 27x + 72$.

In this trinomial a equals 27, b equals 72.

We must, therefore, find two numbers whose sum is 27, and whose product is 72.

The numbers whose product is 72 are 72 and 1, 36 and 2, 24 and 3, 18 and 4, 12 and 6, 9 and 8.

The numbers whose sum gives 27 are 24 and 3.

Hence, $x^2 + 27x + 72 = (x + 24)(x + 3)$.

(2) $a^2 + 3am + 2m^2$.

The numbers whose product is $2m^2$, and whose sum is $3m$, are $2m$ and m. Hence,

$$a^2 + 3am + 2m^2 = (a + 2m)(a + m).$$

EXERCISE 29.

Factor:

1. $x^2 + 4x + 3$.
2. $x^2 + 3x + 2$.
3. $a^2 + 16a + 48$.
4. $x^2 + 9xz + 8z^2$.
5. $x^2 + 126x + 125$.
6. $x^2 + 8x + 12$.
7. $x^2 + 7xy + 12y^2$.
8. $x^2 + 13x + 12$.
9. $a^2 + 40ax + 400x^2$.
10. $a^2 + 74az + 144z^2$.
11. $x^{2a} + 3x^ay^{2b} + 2y^{4b}$.
12. $x^{8m} + 3x^{4m} + 2$.
13. $x^{4+2a} + 5x^{2+a} + 6$.
14. $x^{2(m+n)} + 7x^{m+n}y^{a+b} + 6y^{2(a+b)}$.
15. $a^{2x}b^{2y} + 9a^xb^y + 8$.
16. $x^{-4} + 17x^{-2}y^{-2} + 16y^{-4}$.
17. $x^{-2a} + 12x^{-a}y^{-b} + 20y^{-2b}$.
18. $a^{-8m} + 101a^{-4m} + 100$.
19. $x^{-2(m+n)} + 2x^{-(m+n)} + 1$.
20. $a^{4-2x} + 13a^{2-x} + 42$.

110. XII. **When a trinomial is in the form of $x^2 - ax + b$.**

Substituting $-m$ and $-n$ for $+m$ and $+n$ in 108, we have $x^2 - (m + n)x + mn$.

Now if we put a for $m + n$, and b for mn, we have $x^2 - ax + b$.

111. Hence, *a trinomial of the form of* $x^2 - ax + b$ *can be resolved into two binomial factors:*

The first term of each of which is x.

The sum of the second terms of which is $-a$

The product of the second terms of which is b.

Factor: (1) $x^2 - 13x + 12$.

In this trinomial $-a$ equals -13, and b equals 12.

We must, therefore, find two numbers whose sum is -13, and whose product is $+12$.

112. It is evident that both numbers must be negative, since $+12$ is the product of like factors, and -13 cannot be the algebraic sum of two positive numbers.

The negative numbers whose product is $+12$ are -12 and -1, -6 and -2, -4 and -3.

The numbers whose sum is -13 are -12 and -1.

Hence, $$x^2 - 13x + 12 = (x - 12)(x - 1).$$

(2) $x^2 - 7px + 10p^2$.

The numbers whose product is $+10p^2$, and whose sum is $-7p$, are $-5p$ and $-2p$.

Hence, $$x^2 - 7px + 10p^2 = (x - 5p)(x - 2p).$$

EXERCISE 30.

Factor:

1. $x^2 - 6xz + 8z^2$
2. $x^2 - 12xy + 20y^2$.
3. $x^2 - 30xz + 200z^2$.
4. $x^2 - 24x + 144$.
5. $a^2 - 61ab + 60b^2$.
6. $a^2 - 32ab + 60b^2$.
7. $a^2 - 23ax + 60x^2$.
8. $a^2 - 19a + 60$.
9. $a^2 - 17a + 60$.
10. $a^4 - 11a^2 + 24$.
11. $m^{2a} - 4m^a + 3$.
12. $x^{4a} - 5x^{2a} + 4$.
13. $a^{2x} - 8a^xb^y + 12b^{2y}$.
14. $x^{2a} - 2x^a + 1$.
15. $x^{2a+2b} - 5x^{a+b} + 6$.
16. $x^{-4} - 10x^{2}y^{-3} + 16y^{-6}$.
17. $m^{-4a} - 16m^{-2a} + 48$.
18. $x^{-2}y^{-2} - 12x^{-1}y^{-1} + 27$.
19. $a^{2x-2y} - 3a^{-(x+y)} + 2$.
20. $x^{2m-4} - 9x^{m-2}y^{2} + 18y^{-4}$.

113. XIII. **When a trinomial is in the form of $x^2 + ax - b$.**

Substituting $-n$ for n in (108),

we have, $x^2 + (m - n)x - mn.$

If $m > n$, the quantity $m - n$ is positive.

Now if we put a for $(m - n)$ and b for mn,

we have, $x^2 + ax - b.$

Hence, *a trinomial of the form of* $x^2 + ax - b$ *can be resolved into two binomial factors,*

The first term of each of which is x.

The sum of the second terms of which is $+a$.

The product of the second terms of which is $-b$.

Factor: (1) $x^2 + 5x - 24.$

In this trinomial, a equals 5; $-b$ equals -24.

We must, therefore, find two numbers whose sum is $+5$, and whose product is -24.

114. It is evident that the numbers must differ in algebraic sign, since their product is negative, and that the greater number must be positive since their sum is positive.

The two numbers whose product is -24, the greater being positive, are 24 and -1, 12 and -2, 8 and -3, 6 and -4.

The two numbers whose sum is $+5$ are 8 and -3.

Hence, $x^2 + 5x - 24 = (x + 8)(x - 3).$

(2) $x^2 + 9px - 36p^2.$

The numbers whose product is $-36p^2$ and whose sum is $+9p$ are $+12p$ and $-3p$.

Hence, $x^2 + 9px - 36p^2 = (x + 12p)(x - 3p).$

EXERCISE 31.

Factor:

1. $x^2 + 4x - 21.$
2. $x^2 + 8x - 20.$
3. $x^4 + 10x^2 - 200.$
4. $a^2 + a - 2.$
5. $a^2 + a - 6.$
6. $m^2n^2 + 7mn - 18.$

7. $x^2y^2 + xy - 132$.

8. $x^4 + 5x^2y^2 - 150y^4$.

9. $a^2 + 20ab - 125b^2$.

10. $x^2 + 6xy - 135y^2$.

11. $a^{2x} + a^x - 2$.

12. $x^{2a} + 3x^ay^b - 28y^{2b}$.

13. $x^{4m} + 8x^{2m}y^{2n} - 9y^{4n}$.

14. $a^{2m+2n} + a^{m+n} - 2$.

15. $x^{4(a+b)} + 3x^{2(a+b)}y^{(a\ \ b)} - 54y^{2(a\ \ b)}$.

16. $a^{\ 2} + 2a^{\ 1} - 48$.

17. $a^{-2x} + 6a^{\ x} - 40$.

18. $x^{4-2a} + x^{2\ \ a} - 2$.

19. $m^{-4a} + 4m^{-2a}n^{-2b} - 12n^{-4b}$.

20. $x^2y^{-2} + 5xy^{-1} - 36y^{-2}$.

115. XIV. **When a trinomial is in the form of $x^2 - ax - b$.** Substituting $-m$ for m in (108), we have,

$$x^2 + (-m + n)x - mn.$$

If $m > n$, the quantity $(-m + n)$ is negative, and the sign of the middle term will be negative.

Now if we put $-a$ for $(-m + n)$ and b for mn, we have $x^2 - ax - b$.

116. Hence, *a trinomial of the form of $x^2 - ax - b$ can be resolved into two binomial factors,*

The first term of each of which is x.

The sum of the second terms of which is $-a$.

The product of the second terms of which is $-b$.

Factor: (1) $x^2 - 3x - 54$.

In this trinomial, $-a$ equals -3,
$-b$ equals -54.

We must, therefore, find two numbers whose sum is -3 and whose product is -54.

117. It is evident that the numbers must differ in algebraic sign since their product is negative, and that the greater number must be negative since their sum is negative.

The two numbers whose product is -54, the greater being negative, are -54 and $+1$, -27 and $+2$, -18 and $+3$, -9 and $+6$.

The two numbers whose sum is -3 are -9 and $+6$. Hence:

$$x^2 - 3x - 54 = (x - 9)(x + 6).$$

(2) $x^2 - 5px - 150p^2$.

The two numbers whose product is $-150p^2$ and whose sum is $-5p$ are $-15p$ and $+10p$. Hence:

$$x^2 - 5px - 150p^2 = (x - 15p)(x + 10p).$$

EXERCISE 32.

Factor:

1. $x^2 - x - 2$.
2. $x^2 - 4x - 12$.
3. $a^4 - 2a^2 - 24$.
4. $a^8 - 3a^4 - 54$.
5. $x^2 - 20x - 44$.
6. $x^2 - 15x - 100$.
7. $x^4 - 25x^2 - 150$.
8. $x^2 - 2xy - 48y^2$.
9. $x^2y^2 - xy - 2$.
10. $x^2y^2 - xypq - 2p^2q^2$.
11. $a^{2x} - a^x - 2$.
12. $a^{4x} - 5a^{2x} - 36$.
13. $m^{2n} - 9m^n - 10$.
14. $p^{4a} - 4p^{2a}q^{2b} - 45q^{4b}$.
15. $x^{2(a+b)} - 3x^{(a+b)} - 130$.
16. $a^{-4} - 4a^{-2}b^{-3} - 12b^{-6}$.
17. $x^{-2a} - 16x^{-a}y^{-b} - 36y^{-2b}$.
18. $x^{4-2a} - 25x^{2-a} - 54$.
19. $a^{-4(a+b)} - 21a^{-2(a+b)} - 46$.
20. $a^{-4}b^{-4a} - 2a^{-2}b^{-2a} - 3$.

118. XV. **When a trinomial is in the form of $ax^2 + bx + c$.**

119. From multiplication we have,

$$(mx + p)(nx + q) = mnx^2 + (mq + np)x + pq.$$

In this trinomial m, n, p, and q represent any quantities whatsoever.

Now, if we put $a = mn$,

$b = np + mq$,

$c = pq$,

we have, $ax^2 + bx + c$.

By changing the sign of p and q, and substituting in XI., XII., XIII., XIV., we have the resulting trinomials:

$$ax^2 + bx + c, \qquad ax^2 + bx - c,$$
$$ax^2 - bx + c, \qquad ax^2 - bx - c.$$

120. Therefore we may write the general trinomial,

$$ax^2 \pm bx \pm c,$$

as representing the four possible cases.

Considering,

$$ax^2 + bx + c = mnx^2 + (np + mq)x + pq,$$
$$= (mx + p)(nx + q),$$

we see,

121. I. *That a, the coefficient of x^2, is the product of two numbers, m and n.*

II. *That c, the third term, is the product of two numbers, p and q.*

III. *That b, the coefficient of x, is the sum of two quantities, each of which is the product of the coefficient of the first term of one of the binomial factors with the second term of the other.*

122. If the sign of the third term is positive, the signs of the second terms of the binomial factors may be positive or negative, and this is determined by the sign of the middle term.

If the sign of the third term is negative, the signs of the second terms of the binomial factors are unlike.

This law of signs may be verified by XI., XII., XIII., XIV.

123. Hence:

A trinomial of the form of $ax^2 \pm bx \pm c$ may be resolved into two binomial factors,

The product of the first terms of which is ax^2,

The product of the second terms of which is $\pm c$, and

The sum of the products of the first term of each with the second term of the other is $\pm bx$.

Factor: (1) $6x^2 + 19x + 10$.

In this trinomial, $a = 6$, and $ax^2 = 6x^2$;

$+b = 19$, and $+bx = 19x$;

$+c = 10$.

The signs of the second terms of binomial factors are like and positive.

The two numbers whose product is $6x^2$, are $6x$ and x, $3x$ and $2x$.

The two numbers whose product is 10, are 10 and 1, 5 and 2.

The sum of the products $3x$ times 5 and $2x$ times 2 equals $19x$.

Therefore, $(3x + 2)(2x + 5) = 6x^2 + 19x + 10$.

(2) $6x^2 - 19x + 10$.

In this trinomial, $a = 6$, and $ax^2 = 6x^2$;

$-b = -19$, and $-bx = -19x$;

$c = 10$.

The signs of the second terms of the binomial factors are like and negative.

The numbers whose product is $6x^2$, are $6x$ and x, $3x$ and $2x$.

The negative numbers whose product is 10, are -10 and -1, -5 and -2.

The sum of the products $3x$ times -5 and $2x$ times -2 is $-19x$.

Therefore, $(3x - 2)(2x - 5) = 6x^2 - 19x + 10$.

(3) $6x^2 + 4x - 10$.

In this trinomial, $a = 6$, and $ax^2 = 6x^2$;

$b = 4$, and $bx = 4x$;

$-c = -10$.

The signs of the second terms of the binomial factors are unlike.

The two numbers whose product is $6x^2$ are $6x$ and x, $3x$ and $2x$.

The two numbers whose product is -10 are $+10$ and -1, -10 and $+1$, $+5$ and -2, -5 and $+2$.

The sum of the products $2x$ times $+5$ and $3x$ times -2 is $+4x$.

Therefore, $(3x+5)(2x-2)=6x^2+4x-10$.

(4) $6x^2-4x-10$.

In this trinomial, $a=6$, and $ax^2=6x^2$;

$$-b=-4, \text{ and } -bx=-4x;$$
$$-c=-10.$$

The signs of the second terms of the binomial factors are unlike.

The two numbers whose product is $6x^2$ are $6x$ and x, $3x$ and $2x$.

The two numbers whose product is -10, are $+10$ and -1, -10 and $+1$, $+5$ and -2, -5 and $+2$.

The sum of the products $2x$ times -5 and $3x$ times $+2$ is $-4x$.

Therefore, $(3x-5)(2x+2)=6x^2-4x-10$.

EXERCISE 33.

Factor:

1. $12x^2+7x+1$.
2. $10x^2+19x+6$.
3. $50x^2+15ax+a^2$.
4. $18x^2-39x+15$.
5. $3x^2-36x+60$.
6. $2x^2-3ax+a^2$.
7. $12x^2+11ax-36a^2$.
8. $3a^2x^2+ax-4$.
9. $24a^2-18a-12$.
10. $18x^2-52x-6$.
11. $20a^{2x}+9a^x+1$.
12. $6x^{4a}+13x^{2a}+6$.
13. $2a^{2m}+3a^mb^n+b^{2n}$.
14. $14x^{2p}-34x^p+12$.
15. $16x^{8a}-10x^{4a}y^b+y^{2b}$.
16. $2x^{-2}-3x^{-1}+1$.
17. $18a^{-4}+19a^{-2}-12$.
18. $6x^{2(a+b)}+x^{a+b}-2$.
19. $5x^{2a-4}-4x^{a-2}a-a^2$.
20. $4x^{-2}y^{-2}-7x^{-1}y^{-1}a-2a^2$.

124. XVI. **When a binomial is in the form of $a^n - b^n$.**

125. When n is even, the expression can be reduced to the form of $a^2 - b^2$, and factored according to VIII.

126. When n is odd.

We have, by division,

$$\frac{a^n - b^n}{a - b} = a^{n-1} + a^{n-2}b + a^{n-3}b^2 \cdots ab^{n-2} + b^{n-1}.$$

Hence, we see:

127. *A binomial of the form of $a^n - b^n$ can be resolved into two factors when n is odd.*

One of which is $a - b$.

The other of which is $a^{n-1} + a^{n-2}b + a^{n-3}b^2$, and so on to n terms.

128. The last term, since there are n terms, has the form $a^{n-n}b^{n-1} = b^{n-1}$.

Factor:

(1) $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.

(2) $27a^6 - 1 = (3a^2)^3 - 1$
$\qquad = (3a^2 - 1)(9a^4 + 3a^2 + 1)$.

(3) $a^5 - x^5 = (a - x)(a^4 + a^3x + a^2x^2 + ax^3 + x^4)$.

(4) $a^5 - 32b^5 = a^5 - (2b)^5$
$\qquad = (a - 2b)(a^4 + a^3(2b) + a^2(2b)^2 + a(2b)^3 + (2b)^4)$
$\qquad = (a - b)(a^4 + 2a^3b + 4a^2b^2 + 8ab^3 + 16b^4)$.

(5) $$(a-b)^3 - (a+b)^3 = \left\{\begin{array}{l}(a - b) - (a + b) \qquad \text{and} \\ (a - b)^2 + (a - b)(a + b) + (a + b)^2\end{array}\right.$$
$$= \left\{\begin{array}{l}(a - b - a - b) \qquad \text{and} \\ (a^2 - 2ab + b^2) + (a^2 - b^2) + (a^2 + 2ab + b^2)\end{array}\right.$$
$$= (-2b)(3a^2 + b^2).$$

EXERCISE 34.

Factor:

1. $x^3 - y^3$.
2. $x^3 - 1$.
3. $8a^3 - b^3$.
4. $27a^3 - 1$.
5. $8x^6 - y^3$.
6. $64a^6 - 125b^3$.
7. $27x^3y^6 - 1$.
8. $a^9 - b^3$.
9. $(x + y)^3 - z^3$.
10. $(a + x)^3 - (b + x)^3$.
11. $x^{3m} - y^{3n}$.
12. $8x^{6n} - a^3$.
13. $27a^{6m} - b^{3m}$.
14. $125a^3 - 64b^{3x}$.
15. $1000x^{3m} - 1$.
16. $a^{-3} - 1$.
17. $a^{-6} - b^3$.
18. $8a^{-3x} - b^3$.
19. $64x^{-6a} - y^{-3b}$.
20. $125x^{3a\quad 3b} - a^6$.

129. XVII. **When a binomial is in the form of $a^n + b^n$, n being an odd number.**

130. By division we have,

$$\frac{a^n + b^n}{a + b} = a^{n-1} - a^{n-2}b \cdots - ab^{n-2} + b^{n-1}.$$

Hence:

131. *A binomial of the form of $a^n + b^n$, when n is an odd number, can be resolved into two factors,*

One of which is $a + b$;

The other of which is $a^{n-1} - a^{n-2}b + a^{n-3}b^2$, and so on to n terms.

132. The last term, since these are n terms, is $a^{n-n}b^{n-1} = b^{n-1}$.

133. The terms which contain the odd powers of b are negative.

Factor:

(1) $a^3 + b^3$.

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2).$$

(2) $64x^6 + y^3$.

$$\begin{aligned} 64x^6 + y^3 &= (4x^2)^3 + y^3 \\ &= (4x^2 + y)(16x^4 - 4x^2y + y^2). \end{aligned}$$

(3) $x^5 + y^5$.

$$x^5 + y^5 = (x + y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4).$$

(4) $32x^5 + 1$.

$$\begin{aligned} 32x^5 + 1 &= (2x)^5 + 1 \\ &= (2x + 1)(16x^4 - 8x^3 + 4x^2 - 2x + 1). \end{aligned}$$

(5) $(x + y)^3 + z^3$.

$$\begin{aligned} (x + y)^3 + z^3 &= [(x + y) + z][(x + y)^2 - z(x + y) + z^2] \\ &= (x + y + z)(x^2 + 2xy + y^2 - xz - yz + z^2). \end{aligned}$$

EXERCISE 35.

Factor:

1. $x^3 + 1$.
2. $x^3 + 8y^3$.
3. $27a^3 + b^3$.
4. $1000x^6 + y^3$.
5. $32a^5 + b^5$.
6. $1 + 8a^3$.
7. $a^7 + b^7$.
8. $a^{11} + b^{11}$.
9. $64x^6 + y^3$.
10. $243x^5 + 32y^5$.
11. $x^{3a} + 1$.
12. $x^{6a} + y^{3b}$.
13. $8a^{3m} + b^{3n}$.
14. $32x^{5m} + y^5$.
15. $64a^{6x} + b^{3y}$.
16. $a^{-3} + 1$.
17. $a^{-6} + b^{-3}$.
18. $a^{-6x} + b^{-3y}$.
19. $8x^{-3} + 8y^{-6}$.
20. $a^{-3+6x} + b^3$.
21. $(x + y)^3 + (a + b)^3$.

134. XVIII. **When a binomial is in the form of $a^n + b^n$, n being an even number.**

135. From division we find that $a + b$ is not an integral factor of $a^n + b^n$ when n is even.

136. Factor:

(1) $a^2 + b^2$.

$$\begin{aligned} a^2 + b^2 &= a^2 + 2\,ab + b^2 - 2\,ab \\ &= (a+b)^2 - 2\,ab \\ &= (a + b + \sqrt{2\,ab})\,(a + b - \sqrt{2\,ab}) \\ &= (a + \sqrt{2\,ab} + b)\,(a - \sqrt{2\,ab} + b). \end{aligned}$$

(2) $a^4 + b^4$.

$$\begin{aligned} a^4 + b^4 &= a^4 + 2\,a^2b^2 + b^4 - 2\,a^2b^2 \\ &= (a^2 + b^2)^2 - 2\,a^2b^2 \\ &= (a^2 + b^2 + ab\sqrt{2})\,(a^2 + b^2 - ab\sqrt{2}) \\ &= (a^2 + ab\sqrt{2} + b^2)\,(a^2 - ab\sqrt{2} + b^2). \end{aligned}$$

(3) $a^4 + 4$.

$$\begin{aligned} a^4 + 4 &= a^4 + 4\,a^2 + 4 - 4\,a^2 \\ &= (a^2 + 2)^2 - 4\,a^2 \\ &= (a^2 + 2 + 2\,a)\,(a^2 + 2 - 2\,a) \\ &= (a^2 + 2\,a + 2)\,(a^2 - 2\,a + 2). \end{aligned}$$

137. Hence:

A binomial of the form of $a^n + b^n$, n being even, can be resolved into two trinomial factors,

One of which is the square root of a^n, plus the square root of b^n, plus the square root of twice the product of these terms;

The other of which is the square root of a^n, plus the square root of b^n, minus the square root of twice the product of these terms.

EXERCISE 36.

Factor:

1. $x^2 + y^2$.
2. $a^4 + 1$.
3. $x^8 + 4$.
4. $x^4 + 64\,y^4$.
5. $a^4 + 16\,b^4$.
6. $9\,x^8 + 36\,y^4$.
7. $16\,a^{16} + b^{16}$.
8. $25\,x^{32} + 100$.

9. $x^4 + 324 y^4$.

10. $a^8 + 1024 b^4$.

11. $x^4 + 20.25$.

12. $a^8 + 156.25$.

13. $x^{16} + 2500 y^8$.

14. $4 x^{-4} + 1$.

15. $16 a^{8} + 1$.

16. $a^{-4a} + 4$.

17. $64 x^{-4m} + 1$.

18. $2500 x^{-4} + y^{-4m}$.

19. $a^{-2} + b^{-2}$.

20. $x^{-4(a+b)} + 4$.

138. XIX. **When a polynomial is the product of a binomial and a trinomial, or of two trinomials.**

139. When all the terms of both factors are different, the expression may be factored by Cases III. or IV.

Factor:

(1) $a^2 + 2ab + b^2 + ac + bc$.

$$\begin{aligned} a^2 + 2ab + b^2 + ac + bc &= (a+b)^2 + c(a+b) \\ &= (a+b)(a+b) + c(a+b) \\ &= (a+b)(a+b+c). \end{aligned}$$

(2) $x^2 + 2xy + y^2 - xz - yz - 2z^2$.

$$\begin{aligned} & x^2 + 2xy + y^2 - xz - yz - 2z^2 \\ &= (x+y)^2 - (x+y)z - 2z^2 \\ &= (x+y+z)(x+y-2z). \end{aligned}$$

Factored according to XII.

(3) $2a^2 - 5ab + 2b^2 + 7ac - 5bc + 3c^2$.

$$\begin{aligned} & 2a^2 - 5ab + 2b^2 + 7ac - 5bc + 3c^2 \\ &= (2a^2 - 5ab + 2b^2) + (7a - 5b)c + 3c^2 \\ &= (a - 2b)(2a - b) + (7a - 5b)c + 3c^2 \\ &= (a - 2b + 3c)(2a - b + c). \end{aligned}$$

Factored according to XV.

140. Sometimes the device of grouping the terms is convenient in factoring polynomials of this class.

Thus:

(4) $9x^2 + 12xy + 4y^2 + 21xz + 14yz + 12z^2$.

(*a*) Group the terms containing x and y.

$$9x^2 + 12xy + 4y^2 = (3x + 2y)(3x + 2y).$$

(*b*) Group the terms containing x and z.

$$9x^2 + 21xz + 12z^2 = (3x + 3z)(3x + 4z).$$

(*c*) Group the terms containing y and z.

$$4y^2 + 14yz + 12z^2 = (2y + 3z)(2y + 4z).$$

141. *In the three first binomial factors each term of one trinomial occurs twice. In the three second binomial factors each term of the other trinomial occurs twice. Hence the factors*

$$(3x + 2y + 3z)(3x + 2y + 4z).$$

EXERCISE 37.

Factor:

1. $4x^2 + 4xy + y^2 + 10x + 5y$.
2. $x^2 - 2xy + y^2 - ax + ay$.
3. $x^2 + 5xy + 6y^2 - xz - 4yz - 2z^2$.
4. $2x^2 + 5xy - 3y^2 + x - 4y - 1$.
5. $6x^2 + xy - 35y^2 - 5xz - 27yz - 4z^2$.
6. $x^2 - 5xy + 6y^2 + 8xz - 23yz + 7z^2$.
7. $4x^2 - 12xy + 9y^2 - 10x + 15y$.
8. $9x^2 - 12xy + 3y^2 - 2ayz - a^2z^2$.
9. $15x^2 + 16xy + 4y^2 - 2axz - a^2z^2$.
10. $12x^2 - 25xy + 12y^2 + 53x - 52y + 56$.
11. $x^2 + y^2 + z^2 + 2xy + 2xz + 2yz$.
12. $x^2 + y^2 + z^2 - 2xy - 2xz + 2yz$.

142. XIX. **When a polynomial may be factored by the separation of some one of its terms.**

Factor:

(1) $x^3 + 7x^2 + 8x - 16$.

$$\begin{aligned} x^3 + 7x^2 + 8x - 16 &= x^3 + 7x^2 + 12x - 4x - 16 \\ &= x(x^2 + 7x + 12) - 4(x + 4) \\ &= x(x + 4)(x + 3) - 4(x + 4) \\ &= (x + 4)(x^2 + 3x - 4) \\ &= (x + 4)(x + 4)(x - 1). \end{aligned}$$

(2) $x^3 - 8x^2 + 13x - 6$.

$$\begin{aligned} x^3 - 8x^2 + 13x - 6 &= x^3 - 8x^2 + 12x + x - 6 \\ &= (x^3 - 8x^2 + 12x) + (x - 6) \\ &= x(x^2 - 8x + 12) + (x - 6) \\ &= x(x - 6)(x - 2) + (x - 6) \\ &= (x - 6)(x^2 - 2x + 1) \\ &= (x - 6)(x - 1)(x - 1). \end{aligned}$$

(3) $x^3 - 4x^2 + 8$.

$$\begin{aligned} x^3 - 4x^2 + 8 &= x^3 - 2x^2 - 2x^2 + 8 \\ &= x^2(x - 2) - 2(x^2 - 4) \\ &= x^2(x - 2) - 2(x - 2)(x + 2) \\ &= (x - 2)(x^2 - 2x - 4). \end{aligned}$$

EXERCISE 38.

Factor:

1. $x^3 - 4x^2 + x - 2$.
2. $x^3 - 6x^2 + 2x + 15$.
3. $x^3 - 24x + 5$.
4. $x^3 - 8x^2 + 6x + 7$.
5. $x^3 - 4x^2 - 19x - 14$.
6. $x^3 - 10x^2 + 29x - 20$.
7. $x^3 + 8x^2 + 21x + 14$.
8. $3x^3 - 8x^2 + 6x - 1$.
9. $4x^3 - 16x^2 + 13x - 3$.
10. $x^3 - 26x - 5$.

143. XXI. **When a polynomial is in the form of $a^3+b^3+c^3-3abc$.**

Factor:

(1) $x^3+y^3+z^3-3xyz$.

$$\begin{aligned}x^3+y^3+z^3-3xyz &= x^3-xyz+y^3-xyz+z^3-xyz\\ &= x(x^2-yz)+y(y^2-xz)+z(z^2-xy).\end{aligned}$$

Joining the letters outside the parentheses for one factor, and the letters inside the parentheses for another factor, we have the two required factors,

$$(x+y+z)(x^2+y^2+z^2-yz-xz-xy).$$

(2) $8x^3+27y^3+1-18xy$.

$$\begin{aligned}8x^3+27y^3+1-18xy &= 8x^3-6xy+27y^3-6xy+1-6xy\\ &= 2x(4x^2-3y)+3y(9y^2-2x)+1(1-6xy).\end{aligned}$$

Hence the factors,

$$(2x+3y+1)(4x^2+9y^2+1-3y-2x-6xy).$$

(3) $27x^3+64y^3+125z^3-180xyz$.

$$\begin{aligned}&27x^3+64y^3+125z^3-180xyz\\ &= 27x^3-60xyz+64y^3-60xyz+125z^3-60xyz\\ &= 3x(9x^2-20yz)+4y(16y^2-15xz)+5z(25z^2-12xy).\end{aligned}$$

Hence the factors,

$$(3x+4y+5z)(9x^2+16y^2+25z^2-20yz-15xz-12xy).$$

(4) $a^3+27b^3+8-18ab$.

$$\begin{aligned}&a^3+27b^3+8-18ab\\ &= a^3-6ab+27b^3-6ab+8-6ab\\ &= a(a^2-6b)+3b(9b^2-2a)+2(4-3ab).\end{aligned}$$

Hence the factors,

$$(a+3b+2)(a^2+9b^2+4-6b-2a-3ab).$$

EXERCISE 39.

Factor:

1. $a^3 + b^3 + c^3 - 3abc$.
2. $x^3 + y^3 + z^3 - 3xyz$.
3. $x^3 + y^3 + 8 - 6xy$.
4. $8a^3 + 64b^3 + 1 - 24ab$.
5. $1000x^3 + 125y^3 + 8z^3 - 300xyz$.
6. $a^{3p} + b^{3q} + c^{3m} - 3a^pb^qc^m$.
7. $x^{6a} + y^{9b} + z^{12c} - 3x^{2a}y^{3b}z^{4c}$.
8. $a^{-3} + b^{-3} + 1 - 3a^{-1}b^{-1}$.
9. $8a^{-6} + 27b^{-9} + 1 - 18a^{-2}b^{-3}$.
10. $(a+b)^3 + (c+d)^3 + (c-d)^3 - 3(a+b)(c^2-d^2)$.

144. XXII. **When a polynomial is a perfect power of a binomial.**

Factor:

(1) $a^2 + 2ab + b^2$.

$$\begin{aligned} a^2 + 2ab + b^2 &= a^2 + ab + ab + b^2 \\ &= a(a+b) + b(a+b) \\ &= (a+b)(a+b). \end{aligned}$$

(2) $a^3 + 3a^2b + 3ab^2 + b^3$.

$$\begin{aligned} a^3 + 3a^2b + 3ab^2 + b^3 &= (a^3 + b^3) + (3a^2b + 3ab^2) \\ &= (a+b)(a^2 - ab + b^2) + 3ab(a+b) \\ &= (a+b)(a^2 - ab + b^2 + 3ab) \\ &= (a+b)(a^2 + 2ab + b^2) \\ &= (a+b)(a+b)(a+b). \end{aligned}$$

(3) $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$.

$$\begin{aligned} & x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 \\ &= (x^4 + 4x^3y + 3x^2y^2) + (3x^2y^2 + 4xy^3 + y^4) \\ &= x^2(x^2 + 4xy + 3y^2) + y^2(3x^2 + 4xy + y^2) \\ &= x^2(x + 3y)(x + y) + y^2(3x + y)(x + y) \\ &= (x + y)(x^3 + 3x^2y + 3xy^2 + y^3) \\ &= (x + y)(x + y)(x + y)(x + y). \end{aligned}$$

(4) $x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$

$$\begin{aligned} &= (x^5 + y^5) + (5x^4y + 5xy^4) + (10x^3y^2 + 10x^2y^3) \\ &= (x + y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4) \\ &\qquad + 5xy(x + y)(x^2 - xy + y^2) + 10x^2y^2(x + y) \\ &= (x + y)(x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4) \\ &= (x + y)(x + y)(x + y)(x + y)(x + y). \end{aligned}$$

(5) $8x^3 - 12x^2 + 6x - 1$

$$\begin{aligned} &= (8x^3 - 1) - (12x^2 - 6x) \\ &= (2x - 1)(4x^2 + 2x + 1) - 6x(2x - 1) \\ &= (2x-1)(4x^2 - 4x + 1) = (2x - 1)(2x - 1)(2x - 1). \end{aligned}$$

EXERCISE 40.

Factor:

1. $1 + 3x + 3x^2 + x^3$.
2. $a^4 - 4a^3 + 6a^2 - 4a + 1$.
3. $8a^3 + 12a^2 + 6a + 1$.
4. $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$.
5. $a^5 - 5a^4 + 10a^3 - 10a^2 + 5a - 1$.
6. $8a^3 + 36a^2b + 54ab^2 + 27b^3$.
7. $x^{3a} - 3x^{2a} + 3x^a - 1$.
8. $27 + 27a + 9a^2 + a^3$.
9. $x^{-4} + 4x^{3}y^{-1} + 6x^{-2}y^{-2} + 4x^{-1}y^{-3} + y^{-4}$.
10. $a^{6x} - 3a^{-4x} + 3a^{-2x} - 1$.

EXAMPLES FOR PRACTICE.

EXERCISE 41.

1. $x^2 + 7x + 12.$

2. $x^2 - 9x + 20.$

3. $x^2 - x - 20.$

4. $3a^2 + 7ab + 4b^2.$

5. $a^2 - b^4.$

6. $a^5 - ab^4.$

7. $x^6 - y^6.$

8. $4a^2 - x^2 - 2xy - y^2.$

9. $(a - b)^4 - (x - y)^4.$

10. $x^3 - x^2 - x + 1.$

11. $x^4 - y^4 - x + y.$

12. $a^2x^2 - a^2y^2 - ax + ay.$

13. $a^2 - 2ab + b^2 - x^2 - 2xy - y^2.$

14. $a^4 - a^2b^2 + b^4.$

15. $a^4 + a^2 + 1.$

16. $x^6 + y^6.$

17. $x^4 + 4.$

18. $x^6 - y^6.$

19. $4x^4 - 20x^2y^2 + 16y^4.$

20. $27 - a^3.$

21. $64a - a^5.$

22. $x^{12}y^2 + y^{14}.$

23. $1 - x - 2x^2.$

24. $3xy - 5xyz - 3ax + 5axz.$

25. $49 - 25a^2 - 10ab - b^2.$

26. $(a^2 - b^2 - c^2)^2 - 4b^2c^2.$

27. $6x^2 + 16xy + 8y^2.$

28. $6x^2 - 8xy - 8y^2.$

29. $72xy - 36x^2 - 36y^2 + b^2.$

30. $(a + b)^4 - (a - b)^4.$

31. $(x + y)(x^2 - y^2) - (x - y)(x^2 + y^2).$

32. $49x^4 - 14x^2 + 1.$

33. $a^3 + b^3 + 3ab(a + b).$

34. $x^4 - 4x^3 + 6x^2 - 4x + 1.$

35. $27a^3 - 9a^2 + 3a - 1.$

36. $a^3 + b^3 + c^3 - 3abc.$

37. $6a^2 + ab - b^2 - 3ac + 6bc - 9c^2.$

38. $a^2 - 2ab + b^2 + 10a - 10b.$

39. $a^3 + 3a^2 - 4.$

40. $x^5 + y^5 + 5(x^4y + xy^4) + 10(x^2y^3 + x^3y^2).$

41. $a^2 - ab + ab - b^2.$

42. $27a^3 + 64b^3 + 1 - 36ab.$

43. $(x+y)^3 + (x-y)^3 + (x^2+y^2)^3 - 3(x^4 - y^4).$

44. $x^2 + y^2 + z^2 + 2xy + 2xz + 2yz.$

45. $4a^2 + 9b^2 + 16c^2 + 12ab - 16ac - 24bc.$

46. $4x^2 - 4x + 1 - 16y^2 - 8yz - z^2.$

47. $a^2 + b^2.$

48. $a^3 + ab^4.$

49. $a^3 - 26a - 5.$

50. $x^3 + 8x^2 + 21x + 14.$

51. $a^3 - 6a^2 + 2a + 15.$

52. $(x^2 - x - 2) + (x^2 - 3x + 2).$

53. $9a^2 + 1 + 4b^2 + 6a + 12ab + 4b.$

54. $1 - a^2 - 2ab - b^2.$

55. $x^7 + y^7.$

56. $x^2y^5 + y^2x^5.$

57. $x^8 + y^8.$

58. $a^4 - a^2b^2 + b^4.$

59. $a^4 + 7a^2b^2 + 16b^4.$

60. $a^4 - 82a^2b^2 + 81b^4.$

61. $27a^3 + 64b^3 + 108a^2b + 144ab^2.$

62. $(a^2 + b^2 + c^2)^2 - 4a^2b^2.$

63. $(3x^2 - 2xy - 5y^2) + (6x^2 - 9xy - 15y^2).$

64. $81(a+b+c)^2 - 36(x+y)^2.$

65. $x^{-2} + 2x^{-1} + 1.$

66. $a^{4a} + a^{2a} + 1.$

67. $(x+y)^{-2} - (a+b)^{-2}.$

68. $(x+y)^{-2} + 2(x+y)^{-1} + 1.$

69. $a^4 + 16b^4.$

70. $x^2 + (a+b)x + ab.$

71. $x^2 - (a+b)x + ab.$

72. $x^2 - (a-b)x - ab.$

73. $x^2 + (a-b)x - ab.$

74. $x^2 - \frac{5}{6}x + \frac{1}{6}.$

75. $(x+y)^2 + 7(x+y) + 12.$

76. $(x+y)^2 - 11(x+y) + 18.$

77. $(x+y)^2 - 44(x+y)(a+b) + 43(a+b)^2.$

78 $x^{2n} - y^{2n}.$

79. $a^{-2x} - b^{-2x}.$

80. $a^{2n+12} - 1.$

81. $x^{2a+12} - y^{2a+12}.$

82. $x^{3a} + y^{3b} + z^{3c} - 3x^ay^bz^c.$

83. $5(x+y)^2 + 7(x^2 - y^2).$

84. $(a^2 - 2a + 3)^2 - (a^2 - 3a - 4)^2.$

85. $(x+y)^3+(x-y)^3.$ **87.** $a^{3}-3a^{2}+3a^{1}-1.$

86. $a^3+2a^2+4a+8.$ **88.** $x^{-4}-2x^{-2}y^2+y^4.$

89. $16a^4-17a^2b^2+b^4.$

90. $(a+b)^4+(a+b)^2(a-b)^2+(a-b)^4.$

91. $(a^2+b^2)^2-1+(a^2+b^2+1).$

92. $x^2+y^2+z^2-2xy-2xz+2yz.$

93. $(x^4+4x^3+3x^2)+(3x^2+4x+1).$

94. $3(x^2+x)+1+x^3.$

95. $a^{2(m+n)}+2a^{m+n}b+b^2.$

96. $a^3+27b^3+125c^3-45abc.$

97. $x^{32}-y^{16}.$

98. $x^7+x^2y^5.$

99. $(x^3+x^2+x+1)+(x^5+x^4).$

100. $a^{4x}+4.$

101. $9a^4-12a^2b^2+4b^4.$

102. $81a^4-27a^2b^2+9b^4.$

103. $324a^{4x}-108a^{2x}b^{2y}+b^{4y}.$

104. $25x^{-4}-74x^{-2}y^{-2}+49y^{-4}.$

105. $4x^{-4}-5x^{-2}+1.$

CHAPTER VII.

HIGHEST COMMON FACTOR.

145. Common Factors. A common factor of two or more expressions is an expression which will exactly divide each of them.

Thus, $5x$ is a common factor of $15x^2$ and $25ax^2$.

146. Highest Common Factor. The highest common factor of two or more expressions is the common factor of highest degree which will exactly divide each of them.

Thus, $5x^2$ is the highest common factor of $15x^3$ and $25ax^2$.

147. When unity is the highest common factor of two expressions, they are said to be prime to each other.

Thus, a^2, x, 5, having no factor common except unity, are prime to each other.

148. H. C. F. stands for "Highest Common Factor."

149. I. **When the factors can be determined by inspection.**

Find the H. C. F. of the following:

(1) $25ab^2$ and $75ab^2z$.

$$25ab^2 = 5 \times 5 \times a \times b \times b;$$
$$75ab^2z = 5 \times 5 \times 3 \times a \times b \times b \times z;$$
$$\therefore \text{H. C. F. is } 5 \times 5 \times a \times b \times b = 25ab^2.$$

(2) $96m^2n$, $144mn^2$, $1728m^2n^2$.

$$96m^2n = 3 \times 2 \times 2 \times 2 \times 2 \times 2 \times m \times m \times n;$$
$$144mn^2 = 3 \times 3 \times 2 \times 2 \times 2 \times 2 \times m \times n \times n;$$
$$1728m^2n^2 = 3 \times 3 \times 3 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times m \times m \times n \times n;$$
$$\therefore \text{H. C. F.} = 3 \times 2 \times 2 \times 2 \times 2 \times m \times n = 48mn.$$

(3) $6(ab^2c + bc),\ 12(ab^2 + b)$.

$$6(ab^2c + bc) = 6bc(ab + 1) = 3 \times 2 \times bc(ab + 1),$$
$$12(ab^2 + b) = 12b(ab + 1) = 3 \times 2 \times 2 \times b(ab + 1);$$
$$\therefore \text{H. C. F. is } 3 \times 2 \times b(ab + 1) = 6b(ab + 1).$$

(4) $a^2 + 2ab + b^2,\ a^3 + b^3,\ a^2 - b^2$.

$$a^2 + 2ab + b^2 = (a + b)(a + b);$$
$$a^3 + b^3 = (a + b)(a^2 - ab + b^2);$$
$$a^2 - b^2 = (a - b)(a + b);$$
$$\therefore \text{H. C. F. is } a + b.$$

150. Hence, *the product of the common factors of two or more expressions is the H. C. F. of them all.*

EXERCISE 42.

Find the H. C. F.:

1. $12xyz,\ 36xy^2$.
2. $13a^2bcd,\ 39ab^2c^2$.
3. $11m^2,\ 33m^2n,\ 99m^2n^2$.
4. $a^2 - b^2,\ a^2 - 2ab + b^2$.
5. $a + b,\ (a + b)^2,\ a^3 + b^3$.
6. $a^3 - b^3,\ a^5 - b^5$.
7. $144x^2 - 1,\ (12x - 1)^2,\ 1728x^3 - 1$.
8. $x^2 + x - 12,\ x^2 - 6x + 9,\ x^3 - 27$.
9. $x^3 - y^3,\ x^4 + x^2y^2 + y^4$.
10. $5a^2 + 10ab + 5b^2,\ 5(a^2 - b^2),\ 25(a^3 + b^3)$.
11. $1 - x^2,\ 1 - 3x + 3x^2 - x^3$.
12. $x^{2m} - 1,\ x^{2m} + 2x^m + 1$.
13. $x^{\frac{3}{4}} + y^{\frac{3}{4}},\ x^{\frac{1}{2}} + 2y^{\frac{1}{4}}x^{\frac{1}{4}} + y^{\frac{1}{2}}$.
14. $a^{3m+3} + b^{3n+3},\ a^{2m+2} - b^{2n+2}$.
15. $m^{3x} - 1,\ m^{2x} - 1,\ (m^x - 1)^2$.

151. II. **When the factors cannot easily be determined by inspection.**

152. This case is similar to the method employed in Arithmetic for finding the H. C. F. of large numbers, wherein one expression is divided into another, and the remainder obtained from such division divided into the divisor, and so on, until no remainder occurs. The last divisor is the H. C. F.

153. In applying this rule to polynomials, certain modifications are necessary.

The dividend and divisor should be arranged according to the descending powers of a common letter, and the division continued until the exponent of the common letter in the first term of the remainder is less than its exponent in the first term of the divisor.

A common factor should be removed from the term of either expression if the same is not common to the terms of the other, as such factor can be no part of the H. C. F.

When the first term of the dividend does not contain the first term of the divisor, it may be multiplied by any monomial which is not a factor of the terms of the divisor.

Should the first term of the remainder be negative, the signs of each term may be changed.

A factor common to all the terms of both dividend and divisor may be removed, and reserved as one factor of the H. C. F.

(1) Find the G. C. F. of $12a^3-2a^2-7a-3$ and $3a^2-2a-1$.

$$\begin{array}{r|l}
12a^3-2a^2-7a-3 & \underline{3a^2-2a-1} \\
12a^3-8a^2-4a\phantom{{}-3} & \underline{4a+2} \\
\hline
6a^2-3a-3 & \\
\underline{6a^2-4a-2} & \\
a-1 &
\end{array}$$

Here, the exponent of the first term of the remainder $a-1$ is less than the exponent of the first term of the divisor

$3a^2-2a-1$; hence we proceed by dividing the remainder into the divisor.

$$\begin{array}{r|l} 3a^2-2a-1 & \underline{a-1} \\ \underline{3a^2-3a} & \underline{3a+1} \\ a-1 & \\ \underline{a-1} & \end{array}$$

By this second operation, we obtain no remainder; hence $a-1$ is the H. C. F.

(2) Find the H. C. F. of $36x^2-3x-105$ and $15x^2+31x+10$.

As each term of $36x^2-3x-105$ has a common factor 3, we suppress such, and disregard it, as it can be no part of the H. C. F. We have, as a result, $12x^2-x-35$; but as $15x^2$ is not divisible by $12x^2$, we multiply the second expression by 4, which is not a monomial factor of the first expression.

We have,

$$\begin{array}{r|l} 60x^2+124x+40 & \underline{12x^2-x-35} \\ \underline{60x^2-5x-175} & \underline{5} \\ 129x+215 & \end{array}$$

The last remainder has the exponent of the first term, $129x$, less than the exponent of the first term, $12x^2$, of the divisor. It has also a monomial factor 43, common to each of its terms; this we suppress, and divide the result into the divisor.

$$\begin{array}{r|l} 12x^2-x-35 & \underline{3x+5} \\ \underline{12x^2+20x} & \underline{4x-7} \\ -21x-35 & \\ \underline{-21x-35} & \end{array}$$

Hence, $3x+5$ is the H. C. F.

When it is required to find the H. C. F. of three expressions, first find the H. C. F. of two, and then the H. C. F. of this result and the third expression. This second H. C. F. will be the factor required.

Find the H. C. F of $6a^4 + a^3 - a$, $4a^3 - 6a^2 - 4a + 3$, and $2a^3 + a^2 + a - 1$.

$$\begin{array}{r|l}
4a^3 - 6a^2 - 4a + 3 & 2a^3 + a^2 + a - 1 \\ \cline{2-2}
4a^3 + 2a^2 + 2a - 2 & 2 \\ \hline
-8a^2 - 6a + 5 &
\end{array}$$

Change the signs of this last remainder, and divide it into the divisor after multiplying the divisor by 4.

$$\begin{array}{r|l}
8a^3 + 4a^2 + 4a - 4 & 8a^2 + 6a - 5 \\ \cline{2-2}
8a^3 + 6a^2 - 5a \phantom{{}-4} & a \\ \hline
-2a^2 + 9a - 4 &
\end{array}$$

Multiply this last remainder by 4, and continue the division.

$$\begin{array}{r|l}
-8a^2 + 36a - 16 & 8a^2 + 6a - 5 \\ \cline{2-2}
-8a^2 - 6a + 5 & -1 \\ \hline
42a - 21 &
\end{array}$$

Suppress the monomial factor 21 of this last remainder, and divide result into the divisor.

$$\begin{array}{r|l}
8a^2 + 6a - 5 & 2a - 1 \\ \cline{2-2}
8a^2 - 4a \phantom{{}-5} & 4a \\ \hline
10a - 5 &
\end{array}$$

Suppress monomial factor 5, and continue division.

$$\begin{array}{r|l}
2a - 1 & 2a - 1 \\ \cline{2-2}
2a - 1 & 1 \\ \hline
\end{array}$$

∴ $2a - 1$ is the H. C. F. of two of the expressions.

Again, divide $2a - 1$ into $6a^4 + a^3 - a$.

$$\begin{array}{r|l}
6a^4 + a^3 - a & 2a - 1 \\ \cline{2-2}
6a^4 - 3a^3 \phantom{{}-a} & 3a^3 + 2a^2 + a \\ \cline{1-1}
4a^3 - a & \\
4a^3 - 2a^2 & \\ \cline{1-1}
2a^2 - a & \\
2a^2 - a & \\ \cline{1-1}
\end{array}$$

∴ $2a - 1$ is the factor required.

EXERCISE 43.

Find the H. C. F. of the following:

1. $x^2 - xy - 2y^2$ and $x^2 - 4xy + 4y^2$.

2. $a^3 - a^2b + ab^2 - b^3$ and $a^3 - a^2b - ab^2 + b^3$.

3. $3a^3 + 5a^2 - a + 2$ and $a^3 + a^2 - a + 2$.

4. $3x^4 + 14x^2b^2 - 5b^4$ and $6x^4 - 14x^2b^2 + 4b^4$.

5. $4a^5 - 2a^4 + 4a^3 - 27a^2 + 4a - 7$ and $2a^4 + 6a^3 - 19a^2 + 4a - 5$.

6. $m^3 - m^2 - m + 1$, $m^3 - m^2 + m - 1$, and $3m^2 - 2m - 1$.

7. $a^3 + 5a^2 - 9a + 35$, $a^3 - 4a^2 + 9a - 10$, and $a^3 + 2a^2 - 3a + 20$.

8. $2x^3 - 3x^2y - 5xy^2 + 6y^3$, $2x^3 - x^2y - 12xy^2 - 9y^3$, and $2x^3 + 3x^2y - 8xy^2 - 12y^3$.

CHAPTER VIII.

LEAST COMMON MULTIPLE.

154. A **Multiple** of a quantity is the product of the quantity and an entire factor.

155. A **Common Multiple** of two or more quantities is one into which each of them can be exactly divided.

156. The **Least Common Multiple** of two or more quantities is the least quantity into which each of the given quantities can be exactly divided.

L. C. M. stands for "Least Common Multiple."

157. I. **When the quantities can be factored by inspection.**

158. It is evident that the *Multiple* of a quantity contains all the factors of that quantity.

Also, that a *Common Multiple* of two or more quantities contains all the factors of the given quantities.

Also, that the *Least Common Multiple* of two or more quantities contains all the factors of the given quantities and no others.

159. Hence, *if several quantities are resolved into their simplest factors, and each is taken the greatest number of times it occurs in any given expression, the product of these will be the Least Common Multiple.*

(1) Find the L. C. M. of $25a^2b$ and $35a^3b^2$.

$$25a^2b = 5 \times 5 \times a \times a \times b;$$
$$35a^3b^2 = 7 \times 5 \times a \times a \times a \times b \times b.$$

Here 5 occurs twice in one quantity, 7 once, a three times, b twice; $\therefore$ L. C. M. $= 5 \times 5 \times 7 \times a^3 \times b^2 = 175\,a^3b^2$.

(2) Find L. C. M. of $a^2 - b^2$, $(a-b)^2$, and $a^3 - b^3$.

$$\begin{aligned} a^2 - b^2 &= (a-b)(a+b); \\ (a-b)^2 &= (a-b)(a-b); \\ a^3 - b^3 &= (a-b)(a^2+ab+b^2); \\ \therefore \text{L. C. M.} &= (a-b)(a-b)(a+b)(a^2+ab+b^2) \\ &= a^5 - a^2b^3 - a^3b^2 + b^5. \end{aligned}$$

EXERCISE 44.

Find the L. C. M. of the following:

1. $14\,a^2 + 5\,a - 1$, and $6\,a^2 - a - 2$.
2. $12\,a^2 - 28\,a - 24$, and $12\,a^2 - 42\,a - 24$.
3. $x^2 - 3\,xy + 2\,y^2$, and $x^2 - xy - 2\,y^2$.
4. $6\,x^2y^4 - xy^2 - 1$, and $2\,x^2y^4 + 3\,xy^2 - 2$.
5. $a^2 - 5\,am + 6\,m^2$, $2\,a^2 - 7\,am + 3\,m^2$, and $2\,a^2 - 5\,am + 2\,m^2$.
6. $m^3 - 6\,m^2 + 11\,m - 6$, $m^3 + m^2 - 17\,m + 15$, and $m^3 - m^2 - 14\,m + 24$.
7. $6\,x^4 + 5\,x^3y - 6\,x^2y^2$, $4\,x^3 + 4\,x^2y - 3\,xy^2$, and $4\,x^2 - 6\,xy - 18\,y^2$.
8. $a^3 - b^3$, $a^3 + b^3$, and $a^4 + a^2b^2 + b^4$.
9. $x^2 + y^2 + 2\,xy + 2\,xz + 2\,yz + z^2$, and $x^2 + y^2 - z^2 + 2\,xy$.
10. $x^4 + 4$, $x^4 + 4 + x^2$, and $8\,x^4 + 8\,x^6 + 32\,x^2$.

CHAPTER IX.

REDUCTION OF FRACTIONS.

160. A **Fraction** is an indicated division.

Thus, $\frac{a}{b}$ is a fraction, and indicates that a is divided by b.

161. The quantity above the line is called the *numerator;* the quantity below the line is called the *denominator.*

The numerator and denominator are the terms of the fraction.

162. As multiplying both dividend and divisor by the same quantity is equivalent to multiplying the fraction by unity, the value of the fraction is not changed thereby.

Thus, multiplying both terms of $\frac{a}{b}$ by m, we have $\frac{am}{bm}$, but $\frac{m}{m}$ equals unity; $\therefore\ \frac{a}{b}=\frac{am}{bm}$.

163. The **reduction** of a fraction is the process of changing its *form* without changing its *value.*

164. When the numerator and denominator of a fraction have no common factor, the fraction is in its lowest terms.

Hence, *to reduce a fraction to its lowest terms, resolve both numerator and denominator into their simplest factors, and cancel all that are common to both.*

Reduce the following to lowest terms:

$$(1)\quad \frac{34\,a^2b^2c^4}{51\,a^4b^2c}=\frac{17\,a^2b^2c\times 2\,c^3}{17\,a^2b^2c\times 3\,a^2}=\frac{2\,c^3}{3\,a^2}.$$

$$(2)\quad \frac{a^2-b^2}{(a-b)^2}=\frac{(a-b)(a+b)}{(a-b)(a-b)}=\frac{a+b}{a-b}.$$

(3) $\dfrac{a^3+b^3}{a^4+a^2b^2+b^4}=\dfrac{(a+b)(a^2-ab+b^2)}{(a^2+ab+b^2)(a^2-ab+b^2)}=\dfrac{a+b}{a^2+ab+b^2}.$

(4) $\dfrac{a^4+7a^2+12}{a^4+a^2-12}=\dfrac{(a^2+3)(a^2+4)}{(a^2-3)(a^2+4)}=\dfrac{a^2+3}{a^2-3}.$

If the terms of the fraction cannot be readily factored by inspection, find the H. C. F. by the method of division.

Thus the H. C. F. of

$$\frac{2a^3-12a^2+21a-10}{a^4-8a^3+21a^2-20a+4} \text{ is } a-2.$$

Dividing $(a-2)$ into both terms of the fraction, we have

$$\frac{2a^2-8a+5}{a^3-6a^2+9a-2}.$$

EXERCISE 45.

Simplify:

1. $\dfrac{a^2-ab}{a^2-b^2}.$

2. $\dfrac{a^2+2ab+b^2}{a^3+b^3}.$

3. $\dfrac{x^2+9x+20}{x^2+10x+25}.$

4. $\dfrac{2x^2+3x+1}{x^2+5x+4}.$

5. $\dfrac{a^3+b^3}{a^4-b^4}.$

6. $\dfrac{x^2+xy+y^2}{x^4+x^2y^2+y^4}.$

7. $\dfrac{x^6-y^6}{(x^2+xy+y^2)(x^2-xy+y^2)}.$

8. $\dfrac{x^4+4}{x^3+2x^2+2x}.$

9. $\dfrac{a^4-a^2-20}{a^4-9a^2+20}.$

10. $\dfrac{27x^3-1}{81x^4+9x^2+1}.$

11. $\dfrac{3a^2-2a-1}{a^3+a^2-a-1}.$

12. $\dfrac{x^2+2xy+y^2-z^2}{x^2+y^2+z^2+2xy+2xz+2yz}.$

13. $\dfrac{a^2-b^2-c^2-2bc}{a^2+b^2+2bc+2ab+c^2+2ac}.$

14. $\dfrac{(3a^3-3)(3a^3+81)}{9(a^2+a+1)(a^2-3a+9)}$.

15. $\dfrac{x^{2n}-y^{4m}}{(x^n-y^{2m})^2}$.

16. $\dfrac{(x-1)^{2n+2}}{(x-1)^{3n+3}}$.

165. An improper fraction may be reduced to a whole or mixed number by performing the division indicated. If there is a remainder, this should be placed over the denominator and annexed to the integral part of the quotient.

(1) Reduce $\dfrac{a^5-b^5}{a-b}$.

$$\begin{array}{l|l}
a^5-b^5 & \underline{a-b} \\
\underline{a^5-a^4b} & a^4+a^3b+a^2b^2+ab^3+b^4 \\
\quad a^4b-b^5 & \\
\quad \underline{a^4b-a^3b^2} & \\
\qquad a^3b^2-b^5 & \\
\qquad \underline{a^3b^2-a^2b^3} & \\
\qquad\quad a^2b^3-b^5 & \\
\qquad\quad \underline{a^2b^3-ab^4} & \\
\qquad\qquad ab^4-b^5 & \\
\qquad\qquad \underline{ab^4-b^5} &
\end{array}$$

$$\therefore\ \frac{a^5-b^5}{a-b}=a^4+a^3b+a^2b^2+ab^3+b^4.$$

(2) Reduce $\dfrac{a^5-b^5}{a+b}$.

$$\begin{array}{l|l}
a^5-b^5 & \underline{a+b} \\
\underline{a^5+a^4b} & a^4-a^3b+a^2b^2-ab^3+b^4 \\
\quad -a^4b-b^5 & \\
\quad \underline{-a^4b-a^3b^2} & \\
\qquad +a^3b^2-b^5 & \\
\qquad \underline{+a^3b^2+a^2b^3} & \\
\qquad\quad -a^2b^3-b^5 & \\
\qquad\quad \underline{-a^2b^3-ab^4} & \\
\qquad\qquad ab^4-b^5 & \\
\qquad\qquad \underline{ab^4+b^5} & \\
\qquad\qquad\quad -2b^5, &
\end{array}$$

Writing this remainder over the divisor,

we have, $$\frac{-2b^5}{a+b} = -\frac{2b^5}{a+b}.$$

$$\therefore \frac{a^5-b^5}{a+b} = a^4 - a^3b + a^2b^2 - ab^3 + b^4 - \frac{2b^5}{a+b}.$$

EXERCISE 46.

Reduce the following to whole or mixed numbers:

1. $\frac{a^4+2a^2b^2+b^4}{a^2+b^2}$.
2. $\frac{a^3-1}{a^2+a+1}$.
3. $\frac{x^4+x^2+1}{x^2-x+1}$.
4. $\frac{9x^{2m}-1}{3x^m+1}$.
5. $\frac{x^7-y^7}{x-y}$.
6. $\frac{x^{3n}+1}{x^{2n}-1}$.
7. $\frac{36x^2-12x+1}{6x-1}$.
8. $\frac{a^4+4}{a^2+a+2}$.
9. $\frac{x^2-9x+20}{x-4}$.
10. $\frac{5a^3-a^2+5}{5a^2+4a-1}$.
11. $\frac{1-x^3}{x+1}$.
12. $\frac{a^{\frac{2}{3}}-1}{a^{\frac{1}{3}}-1}$.

166. To reduce a mixed number to a fraction:

Multiply the integral part by the denominator, add the numerator, and place the sum over the denominator.

1. Reduce $x+y+\frac{x}{x-y}$ to a fraction.

$$x+y+\frac{x}{x-y} = \frac{x^2-y^2+x}{x-y} = \frac{x^2+x-y^2}{x-y}.$$

2. $a^2-b^2-\frac{a^4-a^2b^2-b^4}{a^2+b^2}$.

$$a^2-b^2-\frac{a^4-a^2b^2-b^4}{a^2+b^2} = \frac{a^4-b^4-a^4+a^2b^2+b^4}{a^2+b^2} = \frac{a^2b^2}{a^2+b^2}.$$

NOTE. — When the sign of a fraction is minus, it produces a change of sign in each term of the numerator.

EXERCISE 47.

Reduce the following mixed numbers to improper fractions:

1. $x+\frac{x+y}{x-y}$.

2. $x-y-\frac{x-y}{x+y}$.

3. $x^2+y^2-\frac{x^3-y^3}{x^2+y^2}$.

4. $3a^2-\frac{1-a^3}{2a}$.

5. $3a+2b+\frac{a^2-b^2}{a}$.

6. $a-3+\frac{1}{a-6}$.

7. $x+6y+\frac{x^2}{x-6y}$.

8. $a^3+x^3-\frac{a^6-x^6+6}{a^3-x^3}$.

9. $a^2+2ab+b^2-\frac{a}{1+2b}$.

10. $x^2+x+1+\frac{10}{x-1}$.

11. $x+y+z-\frac{(x-y)^2-z^2}{x-y-z}$.

12. $9+x-\frac{x+1}{2x-6}$.

13. $3a-2b+c-\frac{3a-b^2+c}{3-5b}$.

14. $a^2+ax+x^2+\frac{2x^2}{a-x}$.

15. $x^4+x^3+x^2+x+1-\frac{x^5+1}{x-1}$.

167. To reduce fractions to equivalent ones having a common denominator.

As the value of a fraction is not changed by multiplying both terms by the same quantity, it is evident that any number of fractions can be reduced to fractions which are equivalent and have a common denominator.

The least common denominator of several fractions is the least common multiple of their denominators.

Hence, in reducing several fractions to their least common denominator, *divide the denominator of each into the least common multiple of their denominators, multiply the corresponding* *numerator by the quotient thus obtained, and place this product over the least common denominator.*

Each fraction, before its reduction to the least common denominator, should be in its lowest terms.

L.C.D. stands for "Least Common Denominator."

(1) Reduce $\frac{1}{a^2+ab}$ and $\frac{1}{ab+b^2}$ to their least common denominator.

$$\frac{1}{a^2+ab}=\frac{1}{a(a+b)},$$

$$\frac{1}{ab+b^2}=\frac{1}{b(a+b)}.$$

$$\therefore\ ab(a+b) \text{ is the L. C. D.}$$

$$ab(a+b) \div a(a+b) \text{ gives } b;$$

$$ab(a+b) \div b(a+b) \text{ gives } a.$$

Multiplying the corresponding numerators by the results, we have the required fractions,

$$\frac{b}{ab(a+b)} \text{ and } \frac{a}{ab(a+b)}.$$

EXERCISE 48.

Reduce the following to fractions with L. C. D.:

1. $\frac{3a+1}{4}, \frac{3x+4}{6}.$

3. $\frac{a+b}{a-b}, \frac{a-b}{a+b}.$

2. $\frac{9-3a}{16b}, \frac{3+5x}{20b^2}.$

4. $\frac{a+b}{a^2+ab+b^2}, \frac{a-b}{a^2-ab+b^2}.$

5. $\frac{3}{(m-1)(m-2)}, \frac{5}{(m-2)(m-3)}.$

6. $\frac{a-x}{a^3-b^3}, \frac{a+x}{a^3+b^3}.$

7. $\frac{3a}{a^2 - 2ab + b^2}, \frac{4a}{a^2 + 2ab + b^2}, \frac{5a}{a^2 - b^2}.$

8. $\frac{1 - m}{(1 + m)(m - 1)}, \frac{m - 1}{(m + 1)(1 - m)}.$

9. $\frac{1}{1 + x}, \frac{2}{1 - x^2}, \frac{3}{1 - x^3}.$

CHAPTER X.

ADDITION AND SUBTRACTION OF FRACTIONS.

168. Addition of Fractions: *Reduce the given fractions to fractions having a common denominator, add the numerators, and place the sum over the common denominator.*

169. Subtraction of Fractions: *Reduce the given fractions to fractions having a common denominator, subtract the numerator of the subtrahend from the numerator of the minuend, and place the remainder over the common denominator.*

(1) Add the following fractions:

$$\frac{1}{x^2-a^2},\ \frac{2}{x^2-2ax+a^2}.$$

The least common denominator of x^2-a^2 and $x^2-2ax+a^2$ is $(x+a)(x-a)(x-a)$; dividing the given denominators into the L. C. D. and multiplying the corresponding numerators by the quotient thus obtained, we have,

$$x-a \text{ and } 2(x+a).$$

Adding these numerators and placing the result over the L. C. D., we have,

$$\frac{3x+a}{(x-a)^2(x+a)}.$$

EXERCISE 49.

Add the following fractions:

1. $\frac{1}{a+b},\ \frac{2}{a-b},\ \frac{3}{a^2-b^2}.$ 2. $\frac{5x-1}{3a},\ \frac{3x-2}{5a^2},\ \frac{6-3x}{10}.$

3. $\dfrac{3}{1-x}, \dfrac{5a}{1-x^3}, \dfrac{6ab}{1-x^2}.$

4. $\dfrac{4a+5}{5a}, \dfrac{3a-7}{3}, \dfrac{9a+5}{12a^2}.$

5. $\dfrac{(a-b)(a+b)}{a^2-1}, \dfrac{(a+b)(a^2+1)}{a+1}.$

6. $\dfrac{a-(b+c)}{(a-b)^2-c^2}, \dfrac{(a-b)-c}{a^2-(b+c)^2}.$

7. $\dfrac{3x-1}{3x+1}, \dfrac{3x+1}{3x-1}, \dfrac{x}{9x^2-1}.$

8. $\dfrac{x^2-xy+y^2}{x^4+x^2y^2+y^4}, \dfrac{x^2+xy+y^2}{x^6-y^6}.$

(1) Subtract $\dfrac{1}{a+b}$ from $\dfrac{4}{a^2-b^2}$.

The L. C. D. of $a+b$ and a^2-b^2 is a^2-b^2.

Dividing the given denominators into the L. C. D., and multiplying the corresponding numerators by the quotient thus obtained, we have $a-b$ and 4.

Subtracting the numerator of the subtrahend from the numerator of the minuend, we have $4-(a-b)$.

Placing this result over the L. C. D., we have $\dfrac{4-a+b}{a^2-b^2}$.

EXERCISE 50.

Perform the subtraction indicated:

1. $\dfrac{a^2}{(x-y)^2}-\dfrac{a}{x-y}.$

2. $\dfrac{m}{m-1}-\dfrac{m}{m+1}.$

3. $\dfrac{a+b}{a^2-2ab+b^2}-\dfrac{a-b}{a^2+2ab+b^2}.$

4. $\dfrac{x-y}{x^3+y^3}-\dfrac{x+y}{x^3-y^3}.$

5. $\dfrac{a^2-ax+x^2}{a^2+ax+x^2}-\dfrac{a-x}{a+x}.$

6. $\dfrac{a+b+c}{a^2-(b+c)^2}-\dfrac{a-b+c}{(a-b)^2-c^2}.$

7. $\dfrac{x}{x+y+z}-\dfrac{x^2}{x^2+y^2+2xy+2xz+2yz+z^2}.$

170. Sometimes the factors of the denominator of a fraction are not arranged in the same order. When this is the case, the work may be simplified by changing the order of certain factors.

It is necessary in such an operation to consider that *changing the sign of both numerator and denominator does not change the sign of the quotient, and that changing the sign of any even number of factors, in either the numerator or denominator, does not change the sign of the quotient.*

Thus, dividing both terms of $\frac{a}{b}$ by -1, we have $\frac{-a}{-b}$.

$$\therefore\ \frac{a}{b}=\frac{-a}{-b}.$$

Again, $$\frac{a-1}{a-2}=\frac{-(a-1)}{-(a-2)}=\frac{1-a}{2-a}.$$

Again, $$a\times b=ab,$$

and $$-a\times -b=ab.$$

Again, $$(a-1)(a-2)=a^2-3a+2.$$

and $-(a-1)\times-(a-2)=(1-a)(2-a)=a^2-3a+2.$

$$\therefore\ a\times b=-a\times -b=ab,$$

and $(a-1)(a-2)=-(a-1)\times-(a-2)=(1-a)(2-a)$
$=(a^2-3a+2).$

Again, $$\frac{1}{a-b}=\frac{-1}{b-a}=-\frac{1}{b-a},$$

and $$+\frac{1-x+x^2}{a-a^2-a^3}=+\frac{-(1-x+x^2)}{-(a-a^2-a^3)}$$

$$=+\frac{-1+x-x^2}{-a+a^2+a^3}=-\frac{1-x+x^2}{a^3+a^2-a};$$

that is, changing the sign of the fraction is equivalent to changing the sign of each term of the numerator.

Perform the operations indicated:

(1) $$\frac{x-1}{(x-2)(x-3)}+\frac{x+1}{(2-x)(3-x)}.$$

$(2-x)(3-x)$ and $(x-2)(x-3)$ give the product x^2-5x+6; that is, $(2-x)(3-x)=-(2-x)$ times $-(3-x)=x^2-5x+6$.

∴ The second fraction may be written

$$\frac{x+1}{(x-2)(x-3)}.$$

Hence, by addition, we have,

$$\frac{x-1}{(x-2)(x-3)}+\frac{x+1}{(x-2)(x-3)}=\frac{2x}{x^2-5x+6}.$$

(2) $$\frac{1}{(a+b)(c-b)}+\frac{1}{(b-c)(a-b)}+\frac{1}{(b-a)(a+b)}.$$

$$\frac{1}{(a+b)(c-b)}=\frac{1}{(a+b)(c-b)}.$$ No change.

$$\frac{1}{(b-c)(a-b)}=\frac{1}{(c-b)(b-a)}.$$

Sign changed in both factors in denominator.

$$\frac{1}{(b-a)(a+b)}=\frac{1}{(b-a)(a+b)}.$$ No change.

Rewriting the fractions with the proper arrangement of terms, we have, therefore,

$$\frac{1}{(a+b)(c-b)}+\frac{1}{(c-b)(b-a)}+\frac{1}{(b-a)(a+b)}.$$

The L. C. D is $(a+b)(c-b)(b-a)$.

Reducing, we have,

$$\frac{b-a}{(a+b)(c-b)(b-a)}+\frac{a+b}{(a+b)(c-b)(b-a)}+\frac{c-b}{(a+b)(c-b)(b-a)}.$$

Adding, we have,

$$\frac{b-a+a+b+c-b}{(a+b)(c-b)(b-a)}=\frac{b+c}{(a+b)(c-b)(b-a)}.$$

EXERCISE 51.

1. $\dfrac{a-1}{(a-2)(a-3)}+\dfrac{a-2}{(a-4)(a-5)}+\dfrac{a-3}{(5-a)(3-a)}.$

2. $\dfrac{x}{(1-x)(x+1)}-\dfrac{x^2}{(x-1)(1+x)}+\dfrac{x^3}{1-x^2}.$

3. $\dfrac{1}{(a-b)(a-c)(a-d)}+\dfrac{1}{(a+b)(c-a)(d-a)}.$

4. $\dfrac{a-b}{(a+b)(a-c)(a-d)}+\dfrac{a-c}{(b+a)(c-a)(d-a)}-\dfrac{a-c}{(a-c)(d-a)}.$

5. $\dfrac{x-1}{x+1}-\dfrac{x+1}{x-1}+\dfrac{x^2}{1-x^2}.$

6. $\dfrac{1}{(x-2)(x-3)}+\dfrac{2}{(x-1)(3-x)}+\dfrac{1}{(x-1)(x-2)}.$

7. $\dfrac{1}{x(x-y)(x-z)}+\dfrac{1}{z(y-x)(y-z)}+\dfrac{1}{z(z-x)(z-y)}.$

8. $\dfrac{x-2}{x^2+12x+20}+\dfrac{x+2}{x^2+8x-20}.$

Express the denominators in factors, and proceed as above.

CHAPTER XI.

MULTIPLICATION OF FRACTIONS.

171. To multiply one fraction by another:

Cancel the factors common to both terms, multiply the numerators together for a new numerator, and the denominators together for a new denominator.

(1) Multiply $\frac{a}{b}$ by $\frac{c}{d}$.

$$\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}.$$

(2) Multiply $\frac{a^2}{b}$ by $\frac{b^2}{c}$.

$$\frac{a^2}{b}\times\frac{b^2}{c}=\frac{a^2b}{c}.$$

(3) Multiply $\frac{a^2-3a+2}{a^2+3a+2}$ by $\frac{a^2+5a+6}{a^2-5a+6}$.

$$\frac{a^2-3a+2}{a^2+3a+2}=\frac{(a-2)(a-1)}{(a+2)(a+1)}.$$

$$\frac{a^2+5a+6}{a^2-5a+6}=\frac{(a+2)(a+3)}{(a-2)(a-3)}.$$

Hence we have,

$$\frac{(a-2)(a-1)}{(a+2)(a+1)}\times\frac{(a+2)(a+3)}{(a-2)(a-3)}=\frac{(a-1)(a+3)}{(a+1)(a-3)}=\frac{a^2+2a-3}{a^2-2a-3}.$$

172. For convenience in the operation, integral expressions may be considered as having unity for their denominator.

Mixed expressions should be reduced to fractional expressions. Each fractional expression should be in lowest terms.

EXERCISE 52.

Multiply together the following fractions:

1. $\frac{abc}{b^2d^2e^2}, \frac{b^3c^2}{ab^2}$.

2. $\frac{3\,ab}{8\,bc}, \frac{16\,b^2c^2}{9\,a^2b^2}$.

3. $\frac{a^2-ab}{x^2-xy}, \frac{x^2+xy}{a^2+ab}$.

4. $\frac{a^2-a^2b+a^2b^2}{b^2-ab^2+a^2b^2}, \frac{x^2-ax^2+a^2x^2}{y^2-by^2+b^2y^2}$.

5. $\frac{a^2-2\,ab+b^2}{x^4+x^2y^2+y^4}, \frac{x^2-xy+y^2}{(a-b)^2}$.

6. $\frac{x^2-7\,x+12}{x^2-9\,x+20}, \frac{x^2-10\,x+25}{x^2-6\,x+9}$.

7. $\frac{(a-x)^3}{x^3-y^3}, \frac{x^2+xy+y^2}{a^2-2\,ax+x^2}$.

8. $\frac{(a-b)^2-1}{(a+b)^2-1}, \frac{a+b+1}{a-b-1}$.

9. $\frac{a^2-4}{x^2+5\,x}, \frac{x^2-25}{a^2+2\,a}$.

10. $\frac{a^4-1}{x^3+1}, \frac{(x+1)^3}{(a^2+1)(x+1)^2}$.

11. $\frac{a^4+4}{x^4+4}, \frac{x^2-2\,x+2}{a^2-2\,a+2}$.

12. $\frac{(a^3-1)(a^3+1)}{x^4+x^2y^2+y^4}, \frac{(a-1)^3(a+1)^3}{(a^2-1)(a^4-2\,a^2+1)}$.

13. $\frac{100\,x^2-20\,xy+y^2}{169\,a^2-26\,ab+b^2}, \frac{26\,a^2-2\,ab}{20\,x^2-2\,xy}$.

14. $\dfrac{a^{2n}-1}{a^{2m}+1}$, $\dfrac{a^{4m}+2a^{2m}+1}{a^{4n}-2a^{2n}+1}$.

15. $\dfrac{x^{2b+4}-1}{x^{2a+2}+1}$, $\dfrac{x^{4a+4}+2x^{2a+2}+1}{x^{4b+8}-1}$.

16. $\dfrac{x^{\frac{2}{3}}-y^{\frac{2}{3}}}{a^{\frac{2}{3}}+b^{\frac{2}{3}}}$, $\dfrac{a^{\frac{4}{3}}-b^{\frac{4}{3}}}{x^{\frac{8}{3}}-y^{\frac{8}{3}}}$.

17. $\dfrac{a^{b}-1}{x^{a}+1}$, $\dfrac{x^{2a}-1}{a^{2b}-1}$, $\dfrac{a^{2b}+2a^{b}+1}{x^{2a}-2x^{a}+1}$.

173. To divide one fraction by another:

Invert the terms of the divisor and proceed as in multiplication.

(1) Divide $\frac{a}{b}$ by $\frac{c}{d}$.

$$\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}=\frac{ad}{bc};$$

hence $\frac{ad}{bc}$ is the required answer.

(2) Divide $\dfrac{a^2-1}{a^2+2a+1}$ by $\dfrac{a-1}{a+1}$.

$$\frac{a^2-1}{a^2+2a+1}\div\frac{a-1}{a+1}=\frac{(a-1)(a+1)}{(a+1)(a+1)}\times\frac{a+1}{a-1}=\frac{a+1}{a+1}=1.$$

EXERCISE 53.

Perform the division indicated:

1. $\dfrac{3a^2}{24b^2}\div\dfrac{3ax}{12by}$.

2. $\dfrac{5a-10b}{18a-30b}\div\dfrac{2a-4b}{3a-5b}$.

3. $\dfrac{a^2-3a-10}{x^2-7x+12}\div\dfrac{a^2-9a+20}{x^2-8x+16}$.

4. $\dfrac{a^4+4}{a^2-1}\div\dfrac{a^2+2a+2}{a^3-1}$.

5. $\dfrac{a^2-9}{a^3-27}\div\dfrac{a^2+6a+9}{a^2+3a+9}$.

6. $\dfrac{a^6-64b^6}{a^6+64b^6} \div \dfrac{a^4+4a^2b^2+16b^4}{a^4-4a^2b^2+16b^4}.$

7. $\dfrac{1-32x^5}{1+32x^5} \div \dfrac{1-2x}{1+2x}.$

8. $\dfrac{81x^2-1}{169x^2+26x+1} \div \dfrac{81x^2-18x+1}{169x^2-1}.$

9. $\dfrac{(a-b)^2-9}{(a+b)^2-9} \div \dfrac{a-b+3}{a+b+3}.$

10. $\dfrac{a^2+x^2-1+2ax}{x^2+y^2-9+2xy} \div \dfrac{a+1+x}{x+3+y}.$

11. $\dfrac{x^2+y^2+2xy+2xz+2yz+z^2}{a^2+b^2-c^2-d^2+2ab+2cd} \div \dfrac{x+y+z}{(a+b)^2-(c-d)^2}.$

12. $\dfrac{a^3+b^3+c^3-3abc}{(a^2+b^2-c^2)^2-4a^2b^2} \div \dfrac{a^2+b^2+c^2-ab-ac-bc}{(a-b-c)(a-b+c)}.$

13. $\dfrac{a^{3m}-1}{x^{3a}+1} \div \dfrac{a^{2m}-1}{x^{2a}-1}.$

14. $\dfrac{a^{-2}+2a^{-1}+1}{x^{-4}-2x^{-2}+1} \div \dfrac{a^{-3}+1}{x^{-4}-1}.$

15. $\dfrac{x^4-17x^2+16}{9a^4-34a^2+25} \div \dfrac{x^2-3x-4}{3a^2+8x+5}.$

Complex Fractions.

174. A complex fraction is one that has a fraction in the numerator or in the denominator, or in both.

The application of the foregoing principles for simple fractions, integral or mixed expressions, is sufficient for the solution of complex fractions.

(1) Simplify $\dfrac{\dfrac{a+b}{a-b} \div \dfrac{a-b}{a+b}}{\dfrac{x+y}{x-y} \div \dfrac{x-y}{x+y}}.$

$$\frac{a+b}{a-b} \div \frac{a-b}{a+b} = \frac{a^2+2ab+b^2}{a^2-2ab+b^2},$$

$$\frac{x+y}{x-y} \div \frac{x-y}{x+y} = \frac{x^2+2xy+y^2}{x^2-2xy+y^2}.$$

$$\therefore \frac{\dfrac{a+b}{a-b} \div \dfrac{a-b}{a+b}}{\dfrac{x+y}{x-y} \div \dfrac{x-y}{x+y}} = \frac{\dfrac{a^2+2ab+b^2}{a^2-2ab+b^2}}{\dfrac{x^2+2xy+y^2}{x^2-2xy+y^2}}$$

$$= \frac{a^2+2ab+b^2}{a^2-2ab+b^2} \times \frac{x^2-2xy+y^2}{x^2+2xy+y^2} = \frac{(a+b)^2(x-y)^2}{(x+y)^2(a-b)^2}.$$

(2) Simplify $\dfrac{\dfrac{x}{x-y} + \dfrac{y}{x+y}}{\dfrac{y}{x-y} - \dfrac{x}{x+y}}$.

$$\frac{x}{x-y} + \frac{y}{x+y} = \frac{x^2+xy+xy-y^2}{x^2-y^2} = \frac{x^2+2xy-y^2}{x^2-y^2},$$

$$\frac{y}{x-y} - \frac{x}{x+y} = \frac{-x^2+xy+xy+y^2}{x^2-y^2} = \frac{-x^2+2xy+y^2}{x^2-y^2}.$$

$$\therefore \frac{\dfrac{x}{x-y} + \dfrac{y}{x+y}}{\dfrac{y}{x-y} - \dfrac{x}{x+y}} = \frac{\dfrac{x^2+2xy-y^2}{x^2-y^2}}{\dfrac{y^2+2xy-x^2}{x^2-y^2}}$$

$$= \frac{x^2+2xy-y^2}{x^2-y^2} \times \frac{x^2-y^2}{y^2+2xy-x^2} = \frac{x^2+2xy-y^2}{y^2+2xy-x^2}.$$

EXERCISE 54.

Simplify the following complex fractions:

1. $\dfrac{1+\dfrac{1}{x+y}}{1-\dfrac{1}{x-y}}$.

2. $\dfrac{a+b+\dfrac{1}{a-b}}{a-b-\dfrac{1}{a+b}}$.

3. $\dfrac{x+y-\dfrac{x+y}{x-y}}{x-y+\dfrac{x-y}{x+y}}.$

4. $\dfrac{\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}}{\dfrac{1}{x+y+z}}.$

5. $\dfrac{\dfrac{a-b}{a+b}-\dfrac{a+b}{a-b}}{\dfrac{1}{a^2-b^2}-\dfrac{1}{(a-b)^2}}.$

6. $\dfrac{\dfrac{a^2+b^2}{b}-a}{\dfrac{1}{b}-\dfrac{1}{a}}\div\dfrac{a^3+b^3}{a^2-b^2}.$

7. $\dfrac{\dfrac{1}{x}+\dfrac{1}{y+z}}{\dfrac{1}{x}-\dfrac{1}{y+z}}\times\left(1+\dfrac{y^2+z^2-x^2}{2yz}\right).$

8. $\dfrac{1+\dfrac{1}{1+x}}{1+\dfrac{1}{1+\dfrac{1}{1+x}}}.$

9. $\dfrac{\dfrac{a^4-a^3b+a^2b^2-ab^3+b^4}{a^4+a^3b+a^2b^2+ab^3+b^4}\times\dfrac{a^5-b^5}{a^5+b^5}}{\dfrac{a^3+b^3+3ab^2+3a^2b}{a^3-b^3+3ab^2-3a^2b}\div\dfrac{a+b}{a-b}}.$

10. $\dfrac{\dfrac{x-2}{x+2}-\dfrac{x+3}{x-3}}{\dfrac{x+3}{x-3}-\dfrac{x-4}{x+4}}.$

11. $\dfrac{1-\left(\dfrac{x-y}{x+y}\right)^2}{\left(\dfrac{x-y}{x+y}\right)^2+1}.$

12. $\dfrac{\left(\dfrac{x+y}{x-y}\right)^2-\left(\dfrac{x-y}{x+y}\right)^2}{\left(\dfrac{x-y}{x+y}\right)^2+\left(\dfrac{x+y}{x-y}\right)^2}.$

13. $\dfrac{\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}}{\dfrac{x}{y}+\dfrac{y}{z}+\dfrac{z}{x}}.$

14. $\left(\dfrac{a}{a+2b}+\dfrac{2b}{a-2b}\right)\div\left(\dfrac{a}{a-2b}-\dfrac{2b}{a+2b}\right).$

15. $$\frac{\frac{a^2+b^2}{b}+a}{\frac{1}{b}+\frac{1}{a}}\times\frac{a^2-b^2}{b^3-a^3}.$$

16. $$\left(\frac{x}{x+y}+\frac{y}{x-y}\right)\left(\frac{x}{x+a}-\frac{y}{y+a}\right).$$

17. $$\frac{1}{x+\frac{1}{x+\frac{1}{x}}}.$$

18. $$\frac{1}{x+1+\frac{1}{x-1+\frac{1}{x+1}}}.$$

19. $$\frac{a+2}{1-a+\frac{1}{2-a+\frac{3}{a}}}.$$

20. $$\frac{\frac{\frac{1}{x}}{1+\frac{1}{x}}+\frac{1-\frac{1}{x}}{\frac{1}{x}}}{\frac{\frac{1}{x}}{1+\frac{1}{x}}-\frac{1-\frac{1}{x}}{\frac{1}{x}}}.$$

21. $$\left(\frac{x^2-(y-z)^2}{(x+z)^2-y^2}\right)\left(\frac{y^2-(z-x)^2}{(x+y)^2-z^2}\right)\left(\frac{z^2-(x-y)^2}{(y+z)^2-x^2}\right).$$

CHAPTER XII.

FRACTIONAL EQUATIONS.

Solve the equation:

(1) $$\frac{x}{2}+\frac{x}{3}=\frac{x}{4}+35.$$

Clear the equation of fractions by multiplying each term by the L. C. M. of the denominators,

$$6x+4x=3x+420.$$

Transpose the unknown quantities and combine,

$$7x=420.$$

Divide both numbers by 7, and we have

$$x=60.$$

Substituting the value of x in the equation, we have

$$\frac{60}{2}+\frac{60}{3}=\frac{60}{4}+35.$$

The identity $50=50.$

Solve the equation:

(2) $$\frac{3x+10}{5}-\frac{6x-15}{3}=\frac{12x+15}{25}-\frac{3x}{5}.$$

Multiplying each term by 75, the L. C. M. of the given denominators, we have

$$45x+150-(150x-375)=36x+45-45x.$$

Removing parenthesis,

$$45x+150-150x+375=36x+45-45x.$$

Transposing,

$$45x-150x-36x+45x=45-150-375.$$

Combining, $96x=480.$

Dividing by 96, $x=5.$

175. If the numerator of a fraction is a polynomial, and the sign of the fraction negative, it will be found convenient on removing the denominator to enclose the numerator in a parenthesis; if this is not done, *each sign of the numerator should be changed.*

176. Hence, to solve a simple fractional equation:

Clear the equation of fractions by multiplying each term by the L. C. M. of the denominators; transpose the unknown terms to the first member, and the known terms to the second member of the equation; unite the similar terms, and divide by the coefficient of the unknown quantity.

EXERCISE 55.

1. $10x+\frac{x-1}{4}=51.$

2. $x-\frac{x-1}{8}=8.$

3. $5+3x+\frac{9x-16}{2}=4x+3.$

4. $\frac{10x+3}{3}-\frac{6x-7}{2}=10x-10.$

5. $\frac{5-2x}{4}+2-x=-\frac{6x-8}{2}.$

6. $\frac{x+9}{11}+\frac{x-2}{5}=\frac{x+5}{7}.$

7. $\frac{x+2}{5}+\frac{x-1}{7}=\frac{x-2}{2}.$

8. $\frac{x+2}{4}-\frac{x-2}{3}-\frac{x+5}{7}=0.$

9. $\frac{2x}{7}+4-x=-\frac{x-1}{6}.$

10. $\frac{2x-3}{4}-\frac{x-1}{8}=\frac{x}{6}.$

11. $\frac{x-2}{24}+\frac{x-5}{6}=\frac{x-4}{5}.$

12. $\frac{2}{3x}+\frac{2}{4x}=2-\frac{10}{x}.$

13. $3-\frac{1}{3}=\frac{42}{x}-2.$

14. $\frac{6}{7}+\frac{12}{x}=3-\frac{3x-3}{6x}$.
15. $\frac{2}{3}+\frac{35}{3x}=\frac{10}{3}-\frac{5x-10}{9x}$.

16. $\frac{7x+9}{4}-7-x=-\frac{2x-1}{9}$.

17. $\frac{3x+5}{7}-\frac{2x+7}{3}=\frac{3x-50}{5}$.

18. $\frac{12}{x-1}=\frac{10}{x-2}$.
21. $\frac{x}{x-1}-2+\frac{x+1}{x}=0$.

19. $\frac{x+4}{x+3}+\frac{x+2}{x+1}=2$.
22. $x-1=\frac{x^2+3}{x+2}$.

20. $\frac{2x-8}{x-5}-4=-\frac{2x-4}{x-3}$.
23. $\frac{15}{10x-25}=\frac{5}{x}+\frac{60}{2x^2-5x}$.

24. $\frac{7}{x-1}-\frac{6x+1}{x+1}=-\frac{3+6x^2}{x^2-1}$.

25. $\frac{74}{(x+3)(x+2)}=\frac{14}{x+3}+\frac{8}{x+2}$.

26. $\frac{1}{4}\left(\frac{4x}{3}+4\right)-\frac{15-4x}{12}=\frac{x}{2}\left(\frac{3}{x}-1\right)$.

27. $\frac{\frac{3x+6}{2}}{\frac{3x+2}{2}}=-\frac{2x+5}{2x+4}+\frac{x+3}{\frac{x+2}{2}}$.

28. $\frac{6+2x}{1+x}+\frac{2x+10}{x+7}=2+\frac{2(x+1)(x-1)}{(7+x)(1+x)}$.

29. $\frac{18}{x+10}-\frac{9}{x+12}=\frac{12x+33}{(x+10)(x+12)}$.

177. A Literal Equation is one in which some or all of the known members are represented by letters.

(1) Solve $\frac{a}{x+b}-\frac{b}{x-b}=\frac{a^2-b^2}{x^2-b^2}$.

Clear of fractions by multiplying each term by $x^2 - b^2$,

$$ax - ab - bx - b^2 = a^2 - b^2.$$

Cancel $-b^2$ and $-b^2$,

$$ax - bx = a^2 + ab.$$

Factor, $$x(a - b) = a(a + b).$$

Divide by $a - b$, $$\therefore x = \frac{a(a+b)}{a-b}.$$

(2) Solve $ax - \dfrac{bx+1}{x} = \dfrac{a(x^2-1)}{x}.$

Clear of fractions,

$$ax^2 - bx - 1 = ax^2 - a.$$

Transpose and cancel,

$$-bx = -a + 1.$$

Divide by $-b$, $$x = \frac{-a+1}{-b} = \frac{a-1}{b}.$$

$$\therefore x = \frac{a-1}{b}.$$

EXERCISE 56.

Solve the equations:

1. $2a^2 - acx = 3ac - 5abx.$
2. $a - x = (a^2 - x^2)b.$
3. $(a + b)x = (a + b)^2x^2.$
4. $a^4 + a^2 + 1 = x(a^2 - a + 1).$
5. $a - 2ax - 2x - 1 = -a.$
6. $3(c - x) + 3c - x = bx.$
7. $\dfrac{ax}{b} + 1 = c.$
8. $\dfrac{2ax}{a^2 - b^2} = b$
9. $\dfrac{2x+2a}{a-b} - 2a = -\dfrac{2x}{a-b}.$
10. $bx - ax + ax^2 = \dfrac{ax}{b+a}.$
11. $\dfrac{x-b}{a-b} = \dfrac{x+2b}{a+b}.$
12. $\dfrac{a+c}{x} = \dfrac{a+b+c}{x+d}.$
13. $\dfrac{3a+x}{5x} - 1 = \dfrac{6}{5x}.$
14. $\dfrac{ad^2 + ax^2}{2dx} = \dfrac{ac}{2} + \dfrac{ax}{2d}.$

15. $\frac{x-a}{a-b}-\frac{2ax}{a^2-b^2}=\frac{x+a}{a+b}.$

16. $\frac{x+a}{x^2+ax+a^2}-\frac{x-a}{x-ax+a^2}=\frac{a^4}{x(x^4+a^2x^2+a^4)}.$

Problems involving fractional equations.

(1) A man spent one-half of his money, gave away one-third of the remainder, and had twenty-five dollars left.

Find the amount he had at first.

$x =$ his money.

$\frac{x}{2} =$ one-half of his money.

$\frac{1}{3}$ of $\frac{x}{2} = \frac{x}{6} =$ one-third of the remainder.

$$\therefore \frac{x}{2}+\frac{x}{6}+25=x.$$

Clear of fractions,

$$3x+x+150=6x.$$

Transpose and combine the similar terms,

$$-2x=-150.$$

Divide by -2, $x=75.$

∴ The original amount was $75.

EXERCISE 57.

1. The fourth part of a number added to the fifth part makes 9; find the number.

2. The fourth part of a number subtracted from its third part makes 27; find the number.

3. The double of a number added to its ninth part makes 57; find the number.

4. If a pounds of sugar cost b cents, what will one pound cost? What will c pounds cost?

5. Find a number which, if increased by its half, its third, and its fourth part, will amount to 100.

6. A man leaves one-third of his property to his wife, one half of the remainder he leaves to be equally divided between two sons, and the other half to be divided between three daughters. Each daughter received $1500. What was the value of the property? What did each get?

7. A man spends two-fifths of his income for board, two dollars more than one-third of the remainder for clothing; six dollars more than one-half of what then remained for shoes, and had $900 left. What was the income?

8. Divide the number 200 into two parts so that one-half of one part increased by 25 shall equal one-third of the other part.

9. Divide the number 72 into two such parts that one-half of one part decreased by 29 shall equal one-fourth the other part increased by $6\frac{1}{2}$.

10. In a mixture of sugars there are 25 pounds more than one-third of brown and 10 pounds more than one-fifth of white. How many pounds of each are there?

11. In a certain alloy of copper and tin there was more copper than half by 16 pounds, and more tin than one-eighth by 14 pounds. How many pounds were there of each?

12. In a mixture of 100 gallons of wine and water, one-half the wine is to one-third the water as 6 is to 16. How much water must be added to make the proportion of the wine to the water 2 to 13?

13. Find two numbers whose sum is 37, such that the first divided by the second will give a quotient of 1 and a remainder of 7.

14. Divide the number 144 into two such parts that one-fourth the smaller shall be equal to one-sixth the difference between the two parts.

15. A man bought a horse and a cow for 341 dollars. If $\frac{3}{8}$ the price of the horse be subtracted from twice the price of the cow, the remainder will be equal to $\frac{5}{7}$ the price of the cow subtracted from 3 times the price of the horse.

16. If I make a mixture of 72 gallons of wine, water, and milk, taking 3 gallons of milk as often as I take 4 of water and 5 of wine, how many gallons of each have I used?

17. A man pays 100 dollars for rifle, a shot-gun, and revolver. $\frac{1}{2}$ the price of the rifle equals $\frac{1}{3}$ the combined price of the shot-gun and revolver. What did the rifle cost?

18. A can do a piece of work in $3\frac{1}{34}$ days, B in $2\frac{1}{2}$ days, C in $3\frac{1}{3}$ days. If A, B, and C work together, how long will it take to do the work?

19. A tank can be filled with one pipe in 30 minutes, by a second pipe in 40 minutes, by a third in 50 minutes. How long will it take them all to fill it if running together?

20. A tank can be filled by one pipe in $1\frac{1}{2}$ hours and emptied by another pipe in $2\frac{1}{2}$ hours. If both pipes are running, how long will it take to fill the tank?

21. A tank can be filled by one pipe in 25 minutes, by a second in 35 minutes, and emptied by a third in 175 minutes. How long will it take to fill the tank if all the pipes are running together?

22. A runner who does a mile in $5\frac{1}{4}$ minutes is followed by another, who does a mile in 4 minutes 50 seconds, two minutes after starting. How far will they both have proceeded when the second overtakes the first?

23. Two men start at an appointed time, one from Boston, the other from New York. They travel respectively toward New York and Boston. One goes at the rate of $3\frac{1}{2}$ miles an an hour, and rests $\frac{1}{2}$ hour in every six hours. The other goes at the rate of $3\frac{3}{4}$ miles an hour, and rests $\frac{3}{4}$ of an hour

in every eight hours. The distance is 230 miles. In how many hours from time of starting will the men meet? How far has each proceeded?

24. A person rows up-stream at the rate of $3\frac{3}{4}$ miles per hour, and down-stream $4\frac{3}{4}$ miles per hour; if it takes him 10 hours to row up and down, how far up-stream does he go?

25. Two persons, A and B, set out from the same place in opposite directions; A goes three times as far as B less two miles; at the end of 10 hours they are 60 miles apart. How fast does each travel?

26. At what time between one and two o'clock are the hands of a clock together?

27. At what time between five and six o'clock will the hands of a clock be opposite one another?

28. If the minute and second hand on a timing watch are started at twelve o'clock, at what time will they again be together?

29. If the hour and second hand of a timing watch are started at twelve o'clock, at what time will they again be together?

30. On a calendar clock, the day and hour hands are started at twelve o'clock noon January 1st. At what time will they be together again?

31. A timing watch has a second and a fifth-of-a-second hand, each making the complete circuit of the dial in a minute and a second respectively. If started at twelve o'clock, at what time would they again be together?

32. A dog makes five leaps while a cat makes four, but five of the cat's leaps equal six of the dog's leaps. If the cat has 3 leaps the start, how long will it take before the cat is caught if the dog takes five leaps a second?

33. If the radius of a circle were increased one rod, the corresponding area would be increased $9\frac{3}{7}$ square rods. What is the radius and what is the area of both circles?

The value of π is approximately $\frac{22}{7}$.

34. In a certain weight of powder, the saltpetre is 12 pounds more than half, the sulphur 10 pounds less than a third, and the charcoal 6 pounds less than a fourth. How many pounds of each are there?

35. A and B have a sum of money; A takes $6 and one-sixth of what remains, then B takes $9 and one-third of what remains. On counting, they find they have taken equal amounts. What was the sum of money?

36. The length of a rectangle exceeds its breadth by 10 feet. If its length is doubled and its breadth decreased 5 feet, the area will be the same. What were original dimensions?

37. A labourer undertook to perform a piece of work on these conditions: to receive $3 each day he worked, and to forfeit $1.50 each day he was idle; at the end of 40 days he received 75 dollars. How long did he work and how long was he idle?

CHAPTER XIII.

SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE.

If we have an equation containing two unknown quantities, such, for example, as (1) $3x + 4y = 18$, for every value we give to x we shall have a corresponding value of y. In the above equation if

$$\begin{array}{ll} x = 1, & y = 3\frac{3}{4}. \\ x = 2, & y = 3. \\ x = 3, & y = 2\frac{1}{4}. \\ x = 4, & y = 1\frac{1}{2}. \\ x = 5, & y = \frac{3}{4}. \\ x = 6, & y = 0. \end{array}$$

But if we have another equation of like character, such, for example, as (2) $4x + 8y = 32$, and seek to determine what values of x and y satisfy both equations, we shall find but one value for x and one for y.

For multiplying both members of equation (1) by 2, we have,

$$6x + 8y = 36 \qquad (3)$$

Subtracting (2) from (3),

$$\begin{array}{l} 4x + 8y = 32 \qquad (2) \\ \hline 2x \qquad\;\; = 4 \qquad (4) \end{array}$$

Dividing by 2, $\quad x = 2.$

Substituting this value of x in equation (1), we have,

$$6 + 4y = 18.$$

Transposing, $\quad 4y = 12.$

Dividing by 4, $\quad y = 3.$

Hence, if both equations are to be satisfied, x must equal 2, and y must equal 3.

178. Equations of this class which are satisfied by the same values of the unknown quantities are called **simultaneous equations.**

SIMULTANEOUS EQUATIONS.

The process of solution is called *Elimination.*

179. Three well-known methods are given for solving simultaneous equations of the first degree. They are,

I. ADDITION OR SUBTRACTION.
II. SUBSTITUTION.
III. COMPARISON.

I. *Addition or Subtraction.*

Solve:

$$\begin{array}{lrr}
\text{(1)} & \begin{array}{r} 3x - 5y = 10 \\ 9x + y = 46 \\ \hline \end{array} & \begin{array}{r}(1)\\(2)\end{array} \\
\text{Multiply (1) by 3,} & 9x - 15y = 30 & (3) \\
\text{Subtract (2) from (3),} & \begin{array}{r} 9x + y = 46 \\ \hline -16y = -16 \end{array} & \begin{array}{r}(2)\\(4)\end{array} \\
\text{Divide (4) by } -16, & y = 1. & (6) \\
\text{Substitute value of } y \text{ in (1),} & 3x - 5 = 10. & (7) \\
\text{Transpose,} & 3x = 15. & (8) \\
\text{Divide (8) by 3,} & x = 5. &
\end{array}$$

Solve:

$$\begin{array}{lrr}
\text{(2)} & \begin{array}{r} 5x - 4y = 12 \\ 6x + 4y = 32 \\ \hline \end{array} & \begin{array}{r}(1)\\(2)\end{array} \\
\text{Add (1) and (2),} & 11x \quad = 44 & (3) \\
\text{Divide (3) by 11,} & x = 4. & (4) \\
\text{Substitute in (1),} & 20 - 4y = 12. & (5) \\
\text{Transpose,} & -4y = -8. & (6) \\
\text{Divide (6) by } -4, & y = 2. &
\end{array}$$

It is seen from the above solutions, that x or y may be eliminated by addition or subtraction when the coefficients of either letter are the same in both equations.

When the coefficients are not the same, they can be made so by multiplying both members of one or both equations by the quotient obtained from dividing the given coefficients into their L. C. M.

180. Hence, to eliminate by addition or subtraction:

Make the coefficients of one unknown quantity the same in both equations, multiplying each by the proper number.

If the signs of these equal coefficients are unlike, add the equations; if like, subtract one from the other.

EXERCISE 58.

Solve the following equations by addition or subtraction:

1. $\begin{cases} 3x + 3y = 45, \\ 4x - 4y = 28. \end{cases}$

2. $\begin{cases} 15x - 10y = 5, \\ 9y - 12x = 3. \end{cases}$

3. $\begin{cases} 10x - 14y = 66, \\ 11x + 12y = 100. \end{cases}$

4. $\begin{cases} 15x + 21y = 129, \\ 22x + 18y = 138. \end{cases}$

5. $\begin{cases} 6x + y = 26, \\ x + 6y = 16. \end{cases}$

6. $\begin{cases} 19x - 2y = 55, \\ 9x - 7y = 20. \end{cases}$

7. $\begin{cases} 3x - 30y = 15, \\ 2x + 10y = 40. \end{cases}$

8. $\begin{cases} x + 2y = 11, \\ x - y = 5. \end{cases}$

9. $\begin{cases} 10x + 7y = 15, \\ 5x - 7y = 0. \end{cases}$

10. $\begin{cases} \dfrac{x}{3} + \dfrac{3y}{2} = 5, \\ \dfrac{2x}{3} + y = 6. \end{cases}$

11. $\begin{cases} 1.4x - .04y = 4, \\ 2.1x + .06y = 6.6. \end{cases}$

12. $\begin{cases} x - \dfrac{3y}{2} = 3, \\ 2x - \dfrac{8y}{3} = 8. \end{cases}$

II. *Elimination by Substitution.*

Solve:

(1) $$\begin{aligned} x + 3y &= 23 \quad (1)\\ 2x + y &= 16 \quad (2) \end{aligned}$$

In (1) transpose $3y$, $x = 23 - 3y$ (3)

Substitute this value of x in (2),

$$2(23 - 3y) + y = 16. \quad (4)$$

Simplify (4), $46 - 6y + y = 16.$ (5)

Transpose and unite, $-5y = -30.$ (6)

Divide (6) by -5, $y = 6.$

Substitute in (1), $x + 18 = 23.$ (7)

Transpose and unite, $x = 5.$

Solve:

(2) $$\begin{aligned} 3x + 5y &= 29. \quad (1)\\ 5x - y &= 39. \quad (2) \end{aligned}$$

Transpose $5y$ in (1), $3x = 29 - 5y.$ (3)

Divide (3) by 3, $x = \frac{29 - 5y}{3}.$ (4)

Substitute this value of x in (2),

$$5\left(\frac{29 - 5y}{3}\right) - y = 39.$$

Simplify, $145 - 25y - 3y = 117.$

Transpose and unite, $-28y = -28.$

$\therefore\ y = 1.$

Substitute this value of y in (1), $3x = 24.$

$\therefore\ x = 8.$

181. Hence, to eliminate by substitution:

Find in one equation the value of one of the unknown quantities, in terms of the other.

Substitute the value thus obtained in the other equation, and reduce the resulting equation.

EXERCISE 59.

Solve the following equations by substitution:

1. $\begin{cases} 3x - 5y = 10, \\ 2x + 2y = 12. \end{cases}$

2. $\begin{cases} 9x - 29 = -10y, \\ 15x - y = 13. \end{cases}$

3. $\begin{cases} 7x + 9y = 39, \\ 11x + 3y = 39. \end{cases}$

4. $\begin{cases} 13x - 5y = 60, \\ 18x - 7y = 83. \end{cases}$

5. $\begin{cases} 3x - 108 = -3y, \\ x - y = 30. \end{cases}$

6. $\begin{cases} 19x + 18y = 28, \\ x + 10y = 6. \end{cases}$

7. $\begin{cases} 7x - 3y = 22, \\ x + 2y = 8. \end{cases}$

8. $\begin{cases} 5x + 5 = 10y, \\ 8x - 2y = 16. \end{cases}$

9. $\begin{cases} 3x - 7y = 4, \\ 33x + 27y = 234. \end{cases}$

10. $\begin{cases} x + 2y = \frac{4}{3}, \\ 2x - 1 = \frac{1}{2}y. \end{cases}$

11. $\begin{cases} 3x - 2y = -7, \\ 7(x - \frac{1}{3}y) = -\frac{14}{3}. \end{cases}$

12. $\begin{cases} 3(x + 7) = \frac{1}{3}y, \\ 5(x + 4) = y - 1. \end{cases}$

III. *Elimination by Comparison.*

Solve:

(1) $$x + 3y = 27. \qquad (1)$$

$$2x + y = 49. \qquad (2)$$

In (1), $$x = 27 - 3y. \qquad (3)$$

In (2), $$x = \frac{49 - y}{2}. \qquad (4)$$

$$\therefore\ 27 - 3y = \frac{49 - y}{2}.$$

Simplify, $54 - 6y = 49 - y.$

Transpose and unite, $-5y = -5.$

$$\therefore\ y = 1.$$

Substitute in (1), $x + 3 = 27.$

$$\therefore\ x = 24.$$

(2)
$$3x - 5y = 29. \tag{1}$$
$$7x - y = 89. \tag{2}$$

In (1),
$$x = \frac{29 + 5y}{3}. \tag{3}$$

In (2),
$$x = \frac{89 + y}{7}. \tag{4}$$

$$\therefore \frac{29 + 5y}{3} = \frac{89 + y}{7}. \tag{5}$$

Simplify,
$$203 + 35y = 267 + 3y, \tag{6}$$

Transpose and unite,
$$32y = 64. \tag{7}$$
$$\therefore y = 2.$$

Substitute in (1),
$$3x - 10 = 29. \tag{8}$$
$$3x = 39. \tag{9}$$
$$\therefore x = 13.$$

182. Hence, to eliminate by comparison:

Find the value of one of the unknown quantities in terms of the other in each equation; make a new equation from the equal values thus obtained. From this new equation find the value of the one unknown quantity it contains.

EXERCISE 60.

Solve the following equations by comparison.

1. $\begin{cases} x + y = 7, \\ 3x + y = 15. \end{cases}$

2. $\begin{cases} 21x + 23y = 67, \\ 17x + y = 19. \end{cases}$

3. $\begin{cases} 9x - 2y = 79, \\ 8x - 7y = 18. \end{cases}$

4. $\begin{cases} 100x + 99y = 398, \\ 99x + 100y = 398. \end{cases}$

5. $\begin{cases} 33x - 4y = 95, \\ 19x - 7y = 50. \end{cases}$

6. $\begin{cases} 4x - y = 75, \\ x + 5y = 150. \end{cases}$

7. $\begin{cases} \dfrac{x}{2} + \dfrac{y}{3} = 10, \\ \dfrac{x}{5} + y = 17. \end{cases}$

8. $\begin{cases} 49x - 49 = y, \\ 37x - y = 25. \end{cases}$

9. $\begin{cases} .2x + .02y = 22, \\ .7x - .03y = 67. \end{cases}$

10. $\begin{cases} \frac{3}{4}x - y = 16, \\ \frac{1}{2}x + 2y = 12. \end{cases}$

11. $\begin{cases} 18x + .2y = 38, \\ .5x + .6y = 7. \end{cases}$

12. $\begin{cases} \dfrac{x+y}{3} = 3, \\ \dfrac{3x+7y}{5} = 8.6. \end{cases}$

Sometimes the equation appears in complex form which requires simplification before elimination.

Solve:

(1) $\begin{cases} (y+7)(x+5) - 112 = (x+1)(y-9), & (1) \\ 2x - 1 = 3y - 10. & (2) \end{cases}$

From (1) we have, $16x + 4y = 68.$ (3)

From (2) we have, $-2x + 3y = 9.$ (4)

Multiply (4) by 8, $-16x + 24y = 72.$ (5)

Add (3) and (5), $28y = 140.$ (6)

$\therefore y = 5.$

Substitute in (4), $-2x + 15 = 9.$ (7)

Transpose and unite, $2x = 6.$ (8)

$\therefore x = 3.$

EXERCISE 61.

Solve the following equations:

1. $\begin{cases} (x-5)(y+7) = -23 + xy, \\ (x-4)(y+1) = xy - 22. \end{cases}$

2. $\begin{cases} (2x-1)(3y+2) = (x+4)(y+4) - 22, \\ (10x-4)(6y+2) = (4+y)(4+x) + 54. \end{cases}$

3. $\begin{cases} \dfrac{2x+y}{5} = \dfrac{3x-5y+3}{2}, \\ \dfrac{32-x+2y}{8} = \dfrac{3x+2y}{6}. \end{cases}$

4. $$\begin{cases} \dfrac{(y+1)}{y} = \dfrac{(x+7)}{x}, \\ 3x + 1 = 2y + 20. \end{cases}$$

5. $$\begin{cases} \dfrac{3x - 14y}{3} = \dfrac{2x + 2y + 1}{5}, \\ \dfrac{2x}{5} - \dfrac{4y}{5} = 4. \end{cases}$$

6. $$\begin{cases} \dfrac{4(x+1)}{3} = y + 1, \\ \dfrac{3(x-1)}{2} = y - 1. \end{cases}$$

7. $$\begin{cases} \dfrac{2x}{y} = \dfrac{29}{14}, \\ (y + 4x + 6)(4y - 3x - 1) = 4y^2 + 13xy - 12x^2. \end{cases}$$

8. $$\begin{cases} \dfrac{11 + 4x}{5} = \dfrac{7 + 8x}{10} - \dfrac{3x - 6y}{2x - 8}, \\ \dfrac{6y + 9}{4} - \dfrac{13}{4} = \dfrac{3y + 4}{2} - \dfrac{3y + 5x}{4y - 6}. \end{cases}$$

9. $$\begin{cases} \dfrac{x + y}{4xy} = 10 + \dfrac{x - y}{3xy}, \\ \dfrac{x + y}{4} = 10xy - \dfrac{x - y}{3}. \end{cases}$$

10. $$\begin{cases} \dfrac{5x - 3}{2} - \dfrac{3x - 19}{2} = 4 - \dfrac{3y - x}{3}, \\ \dfrac{2x + y}{2} - \dfrac{9x - 7}{8} = \dfrac{3y + 9}{4} - \dfrac{4x + 5y}{16}. \end{cases}$$

11. $$\begin{cases} \dfrac{5x + 7y}{3x + 11} = \dfrac{13}{7}, \\ \dfrac{11x + 27}{7x + 5y} = \dfrac{19}{11}. \end{cases}$$

12. $$\begin{cases} 10[x+9\{y-8(x+7)\}]=6, \\ 5[x+4\{y-3(x+2)\}]=1. \end{cases}$$

13. $$\begin{cases} \dfrac{5y}{6}-\dfrac{4y-19}{3}=\dfrac{x}{6}+\dfrac{20-2y}{3}, \\ \dfrac{x+5y}{6}+5=\dfrac{2y+21}{3}. \end{cases}$$

14. $$\begin{cases} \dfrac{7y+13-5x}{4}+y=2x-\dfrac{3y+2x-16}{3}, \\ x+\dfrac{5y+2x}{6}-\dfrac{3x-12+8y}{5}=4-\dfrac{15+2y-4x}{3}. \end{cases}$$

15. $$\begin{cases} \dfrac{29-x}{6}\div\left(20-\dfrac{4x+5y}{9}\right)=\dfrac{1}{3}, \\ x-\dfrac{3x+4y}{7}-\dfrac{9x-3y-1}{13}=2y-x-16. \end{cases}$$

SIMULTANEOUS EQUATIONS WITH LITERAL COEFFICIENTS.

The letters a', a'', a''' are read *a prime, a second,* and *a third,* respectively. The letters a_1, a_2, a_3 are read *a sub one, a sub two, a sub three,* respectively.

Solve:

(1) $$ax+by=c, \tag{1}$$

$$bx+ay=c'. \tag{2}$$

Multiplying (1) by b and (2) by a, we have,

$$abx+b^2y=bc, \tag{3}$$

and $$abx+a^2y=ac'. \tag{4}$$

Subtracting (4) from (3),

we have, $$b^2y-a^2y=bc-ac', \tag{5}$$

By factoring, $$y(b^2-a^2)=bc-ac'. \tag{6}$$

$$\therefore\ y=\frac{bc-ac'}{b^2-a^2}. \tag{7}$$

Again, multiplying (1) by a, and (2) by b, we have,

$$a^2x + aby = ac, \tag{8}$$

$$b^2x + aby = bc'. \tag{9}$$

Subtracting (9) from (8), we have,

$$a^2x - b^2x = ac - bc'.$$

$$\therefore\ x = \frac{ac - bc'}{a^2 - b^2}.$$

EXERCISE 62.

Solve the following equations:

1. $\begin{cases} ax + by = m', \\ a^2x + b^2y = m''. \end{cases}$

2. $\begin{cases} 2\,ax + y = a', \\ 2\,bx - y = c'. \end{cases}$

3. $\begin{cases} mx + py = a + b, \\ m'x - p'y = a - b. \end{cases}$

4. $\begin{cases} ax + py = r, \\ a^2x + py = r'. \end{cases}$

5. $\begin{cases} \dfrac{x}{b} - \dfrac{y}{c} = a', \\ \dfrac{x}{a} + \dfrac{y}{b} = c'. \end{cases}$

6. $\begin{cases} \dfrac{abx}{c} + \dfrac{bcy}{a} = m, \\ \dfrac{bcx}{a} - \dfrac{aby}{c} = m'. \end{cases}$

7. $\begin{cases} mx - ny = 0, \\ x - y = d. \end{cases}$

8. $\begin{cases} bdx + cdy = e, \\ afx + cgy = h. \end{cases}$

9. $\begin{cases} m - \dfrac{1}{y} = \dfrac{1}{x}, \\ \dfrac{1}{y} - \dfrac{1}{x} = -n. \end{cases}$

10. $\begin{cases} mx + m'y = p, \\ px + p'y = q. \end{cases}$

11. $\begin{cases} n^2x + y = n^2, \\ x + n^2y = n. \end{cases}$

12. $\begin{cases} \dfrac{x}{a} + \dfrac{y}{b} = \dfrac{1}{c}, \\ \dfrac{x}{p} - \dfrac{y}{q} = m. \end{cases}$

13. $\begin{cases} c(c - x) = b(x + y + c), \\ c(y - b) = by - b^2 + cx. \end{cases}$

14. $\begin{cases} c(x + y) - b(x - y) = 2\,c, \\ c(x - y) - b(x + y) = 2b. \end{cases}$

15. $$\begin{cases}(p+q)x-(p-q)y=1,\\(p-q)x+(p+q)y=1.\end{cases}$$

16. $$\begin{cases}\dfrac{a}{x}-\dfrac{b}{y}=c,\\[2ex]\dfrac{m}{x}-\dfrac{n}{y}=p.\end{cases}$$

17. $$\begin{cases}ax-2a^2=-2b^2-by,\\[1ex]\dfrac{y}{a-b}=\dfrac{x}{a+b}+\dfrac{a+b}{ab}.\end{cases}$$

18. $$\begin{cases}\dfrac{x}{m-a}=1-\dfrac{y}{m-b},\\[2ex]\dfrac{x}{n-a}=1-\dfrac{y}{n-b}.\end{cases}$$

183. If three or more simultaneous equations occur, involving three or more unknown quantities, one of the unknown quantities must be eliminated from two or more independent pairs of equations, and a second unknown quantity from the resulting equations, and so on until one unknown quantity only remains.

Each equation must be used in the process of elimination.

184. If the unknown quantities occur with different coefficients, or unlike signs in the several equations, there must be as many equations as there are unknown quantities.

Solve:

(1) $$\begin{cases}x+2y+3z=14, & (1)\\ 2x+3y+4z=20, & (2)\\ 3x+4y+8z=35. & (3)\end{cases}$$

From (1) and (2), we have,

$$\begin{array}{r}2x+4y+6z=28\\ \underline{2x+3y+4z=20}\\ y+2z=\ 8\end{array}\qquad (4)$$

From (2) and (3), we have,

$$\begin{array}{r}6x+9y+12z=\quad 60\\ \underline{6x+8y+16z=\quad 70}\\ y-4z\ =-10\end{array}\qquad (5)$$

From (4) and (5), we have,

$$\begin{aligned} y+2z &= 8 \\ y-4z &= -10 \\ \hline 6z &= 18 \qquad (6) \\ \therefore z &= 3, \\ \therefore y &= 2, \\ \therefore x &= 1. \end{aligned}$$

Solve:

(2)
$$\begin{cases} y+z+u-x=a, & (1) \\ z+u+x-y=b, & (2) \\ u+x+y-z=c, & (3) \\ x+y+z-u=d. & (4) \end{cases}$$

Adding the four equations together and dividing by 2, we have,

$$x+y+z+u=\frac{a+b+c+d}{2}. \qquad (5)$$

Letting $\frac{a+b+c+d}{2}=m,$

and subtracting each equation in turn, we have,

$$x=\frac{m-a}{2},\ y=\frac{m-b}{2},\ z=\frac{m-c}{2},\ u=\frac{m-d}{2}.$$

EXERCISE 63.

Solve the following equations:

1. $\begin{cases} x+2y+3z=14, \\ 3x+3y-z=6, \\ x-y+z=2. \end{cases}$

2. $\begin{cases} 10x+6y-12z=8, \\ 3x-y+2z=8, \\ 2x-4y+4z=4. \end{cases}$

3. $\begin{cases} x+y+z=30, \\ x-y+2z=15, \\ 2x+y-z=35. \end{cases}$

4. $\begin{cases} 3x+y+2z=50, \\ x+2y-3z=15, \\ 2x+2y-3z=25. \end{cases}$

5. $$\begin{cases} 3x + 2y + 4z = 87, \\ 4x + y - z = 4, \\ 5x + 3y - z = 14. \end{cases}$$

6. $$\begin{cases} 3x - 2y - z = 0, \\ 5x - 4y + z = 22, \\ 7x + y - z = 77. \end{cases}$$

7. $$\begin{cases} 5x - y + 3z = 55, \\ 3x + y - z = 31, \\ x - 4y + 5z = 0. \end{cases}$$

8. $$\begin{cases} 3x + 4y + 2z = 0, \\ 15x + y + z = -1, \\ 17x + 17y + 8z = +1. \end{cases}$$

9. $$\begin{cases} \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{13}{12}, \\ \frac{2}{x} + \frac{3}{y} + \frac{4}{z} = 3, \\ \frac{5}{x} + \frac{2}{y} - \frac{8}{z} = 1\frac{1}{6}. \end{cases}$$

10. $$\begin{cases} \frac{2}{x} - \frac{3}{y} + \frac{4}{z} = 2.9, \\ \frac{5}{x} - \frac{6}{y} - \frac{7}{z} = 10\frac{2}{5}, \\ \frac{9}{y} - \frac{8}{x} + \frac{10}{z} = 14.9. \end{cases}$$

11. $$\begin{cases} \frac{3}{x} - \frac{4}{5y} + \frac{1}{z} = 7\frac{3}{5}, \\ \frac{1}{3x} + \frac{1}{2y} + \frac{2}{z} = 10\frac{1}{6}, \\ \frac{4}{5x} - \frac{1}{2y} + \frac{4}{z} = 16.1. \end{cases}$$

12. $$\begin{cases} \frac{1}{x} + \frac{1}{y} = m, \\ \frac{1}{y} + \frac{1}{z} = n, \\ \frac{1}{z} + \frac{1}{x} = p. \end{cases}$$

13. $$\begin{cases} \frac{1}{x} + \frac{1}{y} - \frac{1}{z} = a, \\ \frac{1}{x} - \frac{1}{y} + \frac{1}{z} = b, \\ -\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = c. \end{cases}$$

14. $$\begin{cases} xy + yz + xz = 9xyz, \\ yz + 2xz - 3xy = -4xyz, \\ 3yz - 2xz + xy = 4xyz. \end{cases}$$

15. $$\begin{cases} \frac{x}{b+c} + \frac{y}{c-a} = a + b, \\ \frac{y}{c+a} + \frac{z}{a-b} = b + c, \\ \frac{z}{a+b} + \frac{x}{b-c} = c + a. \end{cases}$$

16. $$\begin{cases} ax + by - cz = 2xy, \\ -ax + by + cz = 2yz, \\ ax - by + cz = 2xz. \end{cases}$$

17. $$\begin{cases} \frac{x}{m} + \frac{y}{m-1} + \frac{z}{m-2} = 1, \\ \frac{x}{p} + \frac{y}{p-1} + \frac{z}{p-2} = 1, \\ \frac{x}{q} + \frac{y}{q-1} + \frac{z}{q-2} = 1. \end{cases}$$

18. $$\begin{cases} \frac{x}{3}+\frac{y}{5}+\frac{z}{7}+\frac{u}{9}=2800, \\ \frac{x}{5}+\frac{y}{7}+\frac{z}{9}+\frac{u}{11}=2144, \\ \frac{x}{7}+\frac{y}{9}+\frac{z}{11}+\frac{u}{13}=1744, \\ \frac{x}{9}+\frac{y}{11}+\frac{z}{13}+\frac{u}{15}=1472. \end{cases}$$

PROBLEMS INVOLVING TWO OR MORE UNKNOWN QUANTITIES.

185. When it is necessary to employ two or more unknown quantities in the solution of a problem, the conditions of the problem must give as many equations as there are unknown quantities.

(1) The sum of two numbers is 144, and their difference is 142. What are the two numbers?

Solution:

Let $x =$ larger number,

and $y =$ smaller number.

Then $x + y = 144$

and $x - y = 142$

$$\therefore\ 2x = 286$$

$$x = 143.$$

$$\therefore\ y = 1.$$

(2) The sum of three numbers is 30. Twice the sum of the first and second less the third is 45; three times the sum of the first and third less the second is 50. Find the numbers.

Solution:

Let $x =$ 1st number,

$y =$ 2d number,

$z =$ 3d number.

Then

$$\begin{aligned} x+y+z&=30 && (1)\\ 2x+2y-z&=45 && (2)\\ 3x+3z-y&=50 && (3)\end{aligned}$$

Add (1) and (2),

$$3x+3y=75 \qquad (4)$$

Multiply (1) by 3 and subtract (3) from the result,

$$\begin{aligned} 3x+3y+3z&=90\\ 3x-\ \ y+3z&=50\\ \hline 4y\qquad\quad &=40 && (5)\\ \therefore\ y&=10.\end{aligned}$$

Substitute value of y in (4),

$$3x+30=75.$$
$$\therefore\ x=15.$$

Substitute values of x and y in (1),

$$15+10+z=30.$$
$$\therefore\ z=5.$$

Hence, the three numbers are respectively 15, 10, 5.

EXERCISE 64.

1. At the meeting of a certain club there were present 179 members; a motion to adjourn was carried by a majority of 21. How many voted on each side?

2. Nine years ago I was eleven times as old as my brother; now I am twice as old. What are our ages?

3. The sum of two numbers divided by 3 gives a quotient of 11, and the difference between them is 3. What are the numbers?

4. The number 125 is divided into two parts; when the first is divided by the second the quotient is 5, and the remainder is 5. What are the numbers?

5. The number 96 is divided into three parts. The first part divided by the third gives a quotient of 3 and a remainder of 2. The sum of the first and second divided by the third gives a quotient of 5. What are the numbers?

6. If I mix 10 lbs. of white and 7 lbs. of brown sugar together, I can sell the 17 lbs. for $1.48; but if I mix 7 lbs. of white with 11 lbs. of brown, I can sell the mixture for $1.40. What is the price of each?

7. A and B have certain sums of money. If A gives B $25, B will have four times as much as A; but if B gives A $50, A will have four times as much as B. What are the original sums?

8. A grocer sold to a retail customer 30 gallons of syrup and 50 gallons of vinegar for $20, to a wholesale customer he made a reduction of 10 per cent and sold 60 gallons of syrup and 25 gallons of vinegar for $29.25. What was the retail price?

9. The cost of 45 cows and 55 sheep is $2800; the cost of 45 sheep and 55 cows, $3200. What is the cost per head?

10. If I add 8 to the numerator of a fraction, it becomes equal to 1; if I subtract 10 from the denominator it becomes equal to 3. What is the fraction?

11. A sum of money was divided between certain men. It was found that if two were added to the number, the amount each received would be $3 less. A dispute arose, and four men were shut out; it was then found that each would receive $6 more than the original share.

12. A certain fraction becomes $\frac{3}{4}$ if 5 be subtracted from the numerator, but if 19 be added to the numerator it becomes $\frac{3}{2}$. Find the fraction.

13. A certain number consists of two digits. If the number is divided by the sum of the digits, the quotient is 4. If the digits are reversed and increased by 12, the result is equal to twice the original number. Find the number.

14. If a certain number is divided by the sum of its two digits, the quotient is 6, and the remainder 6. If the digits are interchanged and the number thus formed increased by 1, the quotient obtained by dividing by the sum of the digits will be 4. What is the number?

15. A took 30 marbles from B, and found then he had twice as many as B. If B then took 60 from A, they would have had the same number. How many did each have originally?

16. A certain number is expressed by three digits:

The sum of the digits equals 8. The number divided by the sum of the first two digits gives a quotient of 40 and a remainder of 5.

The number divided by the sum of the second and third digits increased by 18 gives a quotient of 5. Find the number.

17. A person rows down stream ten miles and back again in 5 hours. He can row 3 miles against the stream in the same time he can row 4 miles with it. Find the rate of the current; the rate of speed down and up.

18. A and B can do a piece of work in 8 days, B and C in 6 days, A and C in $4\frac{4}{5}$ days. How long will it take each to do it?

19. A cistern has 3 pipes. The first and second can fill it in 20 minutes, the first and third in 12 minutes, the second and third in 10 minutes. How long will it take each to fill it?

20. If I add 5 feet to the length, and two feet to the width, of a rectangular field, I increase its area 100 square feet, but if I add 10 feet to its length and subtract 5 feet from its width, I decrease its area 50 square feet. Find the original dimensions.

21. Divide each of the numbers 80, 90, and 100 into two parts, so that the sum of the smaller parts shall equal 120, the sum of the larger part of 80 and the smaller part of 90 shall equal 90, and the sum of the smaller part of 100 and the larger part of 90 shall equal 100.

22. In a bag containing 60 white, red, and black balls, $\frac{2}{3}$ the number of black equals $\frac{1}{2}$ the number of white plus $\frac{3}{4}$ the number of red balls; and $\frac{1}{5}$ the number of white plus $\frac{1}{4}$ the number of red plus $\frac{1}{5}$ the number of black equals $\frac{1}{2}$ the number of red plus $\frac{1}{10}$ the number of black. Find the number of each.

23. Sixty tons of coal are to be taken from a mine in three carts, A, B, and C, respectively. It may be taken as follows: 20 loads in A, 10 loads in B, and 10 loads in C; or 10 loads in A, 20 loads in B, and 20 loads in C; or 15 loads in A, 14 loads in B, and 18 loads in C. Find the capacity of each cart.

24. A farmer sells 2 bushels of oats, 3 of rye, and 4 of wheat for $ 20; and 3 bushels of oats, 4 of rye, and 5 of wheat for $ 26; and 3 bushels of oats, 5 of rye, and 6 of wheat for $ 31. Find the price per bushel of each.

25. Two men, A and B, run a race of 100 miles. A starts one hour ahead of B, but finds when he reaches the fifty-mile point that B is four miles ahead of him. A then doubles his speed and reaches the 100-mile point two hours and forty minutes ahead of B. Find the rate of each.

CHAPTER XIV.

INEQUALITIES.

186. It is often useful to know which is the greater of two quantities. The expression which signifies that one quantity is greater or less than another is called an **Inequality.**

187. In treating of inequalities, greater and less must be understood in their algebraic sense; that is, a positive quantity is greater, and a negative quantity is less, than zero. From this it follows that if a is greater than b, $a-b$ is positive.

188. Different expressions containing a given letter have their relations changed as different values are assigned to the given letter.

Thus, if $x=1,\ 2x>x^2$;

but if $x=3,\ 2x<x^2$.

189. Certain expressions, however, do not change their relation whatever the value assigned to the given letter. Thus, $2x$ is never greater than x^2+1.

For if $x=1,\ 2x=x^2+1$;

if $x=3,\ 2x<x^2+1$;

if $x=\frac{1}{2},\ 2x<x^2+1$.

If a is greater than b, $a^2-2ab+b^2$ is a positive quantity, and $a^2+b^2>2ab$.

190. If any expression, therefore, can be reduced to the form $a^2+b^2>2ab$, an inequality is shown to exist.

191. The first member of an inequality is the expression to the left of the sign of inequality, the second member is the expression to the right of this sign.

192. Two inequalities are said to ***subsist in the same sense*** when the first member of each is greater or less than the second member of each. Thus, $a > b$ and $b > c$ ***subsist in the same sense.***

193. *If equal quantities be added to or subtracted from both members of an inequality, the relation is not altered.*

For if $\quad a > b,\ a - b$ is positive,

and $\quad a + c - (b + c)$ is positive,

also, $\quad a - c - (b - c)$ is positive.

$$\therefore\ a + c > b + c,$$

and
$$a - c > b - c.$$

From the above it may be inferred that any term may be removed from one member of an inequality to the other *if its sign be changed.*

194. *If the signs of all the terms of an inequality be changed, or if both members be multiplied by the same negative quantity, the sign of inequality must be reversed.*

For, changing all of the signs, or multiplying by a negative quantity, is equivalent to interchanging the members of the inequality.

195. *If both members of an inequality are multiplied or divided by the same positive quantity, the relation is not altered.*

For if $a > b$, $a - b$ is positive, and if m be positive, $m(a - b)$ is positive, and $ma > mb$. Substituting $\frac{1}{m}$ for m, we have $\frac{a}{m} > \frac{b}{m}$.

If $$a > b,\ c > d,\ e > f, \cdots,$$

then $$a + c + e + \cdots > b + d + f + \cdots.$$

For the sum of

$$a - b + c - d + e - f \text{ is positive.}$$

$$\therefore\ a + c + e - (b + d + f) \text{ is positive.}$$

$$\therefore\ a + c + e > b + d + f.$$

If $a > b$, $c > d$, $e > f \cdots$, and all the quantities are positive, it is obvious that $ace > bdf$.

Solve:

(1) If $a > b$, and the letters positive,

$$a^4 - b^4 > 2\,a^3b - 2\,ab^3.$$

Factoring, we have,

$$(a^2 + b^2)(a^2 - b^2) > 2\,ab(a^2 - b^2).$$

Dividing by $a^2 - b^2$, we have,

$$a^2 + b^2 > 2\,ab. \qquad \text{(Art. 190.)}$$

(2) $$\frac{a^2 + b^2 + c^2}{2} > a + b + c - \frac{3}{2}.$$

For, $$a^2 + 1 > 2\,a,$$

$$b^2 + 1 > 2\,b,$$

$$c^2 + 1 > 2\,c.$$

Adding, $$a^2 + b^2 + c^2 + 3 > 2\,a + 2\,b + 2\,c.$$

Dividing by 2, $$\frac{a^2 + b^2 + c^2}{2} + \frac{3}{2} > a + b + c.$$

Transposing $\frac{3}{2}$, $$\frac{a^2 + b^2 + c^2}{2} > a + b + c - \frac{3}{2}.$$

EXERCISE 65.

Solve the following examples:

1. $\frac{3a^2+3b^2+3}{2} > 3ab + \frac{3}{2}$.
2. $x^2 + 3y^2 > 2y(x+y)$.
3. $a^6 + b^6 > 2a^5b - 2a^3b^3 + 2ab^5$.
4. $a + b > \frac{4ab}{a+b}$.
5. $x^2 + y^2 + z^2 > xy + xz + yz$.
6. $am(x^4 - y^4) > 2(amx^3y - amxy^3)$.
7. $x^3 + y^3 + z^3 + y^2x + x^2z + yz^2 > 2(x^2y + xz^2 + y^2z)$.

Find the limit of x in the inequality

(1) $\frac{3x}{4} - \frac{x-1}{2} < 6x - \frac{20x+13}{4}$.

Clearing of fractions,

$$3x - 2x + 2 < 24x - 20x - 13.$$

Transposing and uniting,

$$-3x < -15.$$

$$\therefore\ x > 5.$$

If we have given an inequality and an equation containing two unknown quantities, the limit of each may be found by the process of elimination.

Find the limit of x and y in the following:

(2) $$3x - 4y > 5. \qquad (1)$$

$$7x + 5y = 100. \qquad (2)$$

Multiplying (1) by 7, and (2) by 3, we have,

$$21x - 28y > 35. \qquad (3)$$

$$21x + 15y = 300. \qquad (4)$$

Subtracting, we have,

$$-43y > -265$$

$$\therefore\ y < 6\tfrac{7}{43}.$$

By like process, we find that:

$$x > 9\tfrac{38}{43}.$$

Find the limit of x in the following:

8. $3x - 5 > 34.$

9. $\frac{2}{3}x + \frac{3}{5}x < 40.$

10. $\frac{x}{2} + \frac{x}{3} + \frac{x}{4} > 65.$

11. $6x^2 + 7x - 3 > 6x^2 + 17x - 13.$

12. $\frac{x-4}{8} - 1 > \frac{y-10}{6}.$

13. $10x + 5 > 3x + 33.$

14. $ax - a^2 < ab - bx.$

Find the limits of x and y in the following:

15. $\begin{cases} x + y > 11, \\ 3x - y = 20. \end{cases}$

16. $\begin{cases} 4x - 3y < 15, \\ 4x + y = 23. \end{cases}$

17. $\begin{cases} \frac{x}{2} + \frac{y}{3} > x - \frac{y}{2}, \\ 2x + \frac{y}{3} = 200. \end{cases}$

Find the integral value of x in the following:

18. $\begin{cases} 5x + y > 51, \\ 3x - y < 21. \end{cases}$

19. $\begin{cases} 9x + \frac{5y}{7} > 31, \\ 13x - y < 33. \end{cases}$

CHAPTER XV.

INVOLUTION AND EVOLUTION.

196. Involution. The operation of raising any given expression to any power whose exponent is a positive integer is called **Involution.**

The process of involution is simply one of multiplication in which all of the factors are equal.

To raise a monomial to a given power.

EXAMPLE: Find the value of $(a^2)^5$.

$$(a^2)^5 = a^2 \times a^2 \times a^2 \times a^2 \times a^2 = a^{2+2+2+2+2},$$
$$= a^{10}.$$

Now if, in the given example, we substitute m for 2 and n for 5, we have,

$$(a^m)^n = a^m \times a^m \times a^m \cdots \text{to } n \text{ factors times},$$
$$= a^{m+m+m+m \dots \text{to } n \text{ times}},$$
$$= a^{mn}.$$

By the same law a fraction may be raised to any required power by performing the operation indicated upon both numerator and denominator.

EXAMPLE: Find the value of $\left(-\frac{2a^2}{3b^3}\right)^7$.

$$\left(-\frac{2a^2}{3b^3}\right)^7 = -\frac{2^7(a^2)^7}{3^7(b^3)^7} = -\frac{2^7a^{14}}{3^7b^{21}} = -\frac{128a^{14}}{2187b^{21}}.$$

197. *It is evident from the rule in multiplication that even powers of negative quantities give plus, and odd powers give minus.*

To raise a binomial to a given power.

Example: Find the value of $(a+b)^4$.

By multiplication we have,

$$a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

An examination of this result will give us the following laws:

I. *That the number of terms is one greater than the index of the power.*

II. *That the first term contains only the first term of the binomial with an exponent equal to the index of the power, and in each succeeding term its exponent decreases by* 1.

III. *That the second term of the binomial appears for the first time in the second term of the result with* 1 *as its exponent, and in each succeeding term its exponent increases by* 1.

IV. *That the coefficient of the first term is* 1, *and the coefficient of each following term is found by multiplying the exponent of the first letter and the coefficient of the next preceding term together, and dividing this result by the number of this term.*

Applying the above laws, raise $(x+y)$ to the seventh power. Tabulating results, we have,

Powers of x,	$x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x$
Powers of y,	$y + y^2 + y^3 + y^4 + y^5 + y^6 + y^7$
Coefficients,	$1 + 7 + 21 + 35 + 35 + 21 + 7 + 1$
Multiplying,	$x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4$ $+ 21x^2y^5 + 7xy^6 + y^7$

Again, expand $(x-y)^5$.

Powers of x,	$x^5 + x^4 + x^3 + x^2 + x$
Powers of y,	$-y + y^2 - y^3 + y^4 - y^5$
Coefficients,	$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$
Multiplying,	$x^5 - 5x^4y + 10x^3y^2 - 10x^2y^3 + 5xy^4 - y^5$

198. It will be noticed that each term which contains an odd power of y is negative.

Should the terms of the binomial have coefficients or exponents, the rule holds true.

Example (1): Expand $(2x^3-3y^5)^3$.

Powers of $2x^3$, $(2x^3)^3+(2x^3)^2+(2x^3)$

Powers of $-3y^5$, $-(3y^5)+(3y^5)^2-(3y^5)^3$

Coefficients, $1 \quad 3 \quad 3 \quad 1$

Multiplying, $(2x^3)^3-3(2x^3)^2(3y^5)+3(2x^3)(3y^5)^2-(3y^5)^3$

By reduction, $8x^9-36x^6y^5+54x^3y^{10}-27y^{15}$

By the same method any polynomial may be raised to any power.

Example (2): Expand $(a+b-c^2)^3$.

Reducing $(a+b-c^2)^3$ to the form of a binomial, we have, $\{(a+b)-c^2\}^3$.

Performing the operation indicated, we have,

Powers of $(a+b)$, $(a+b)^3+(a+b)^2+(a+b)$

Powers of $-c^2$, $-(c^2)+(c^2)^2-(c^2)^3$

Coefficients, $1 \quad 3 \quad 3 \quad 1$

Multiplying, $(a+b)^3-3(a+b)^2c^2+3(a+b)(c^2)^2-(c^2)^3$

Reducing,

$$a^3+3a^2b+3ab^2+b^3-3a^2c^2-6abc^2-3b^2c^2+3ac^4+3bc^4-c^6.$$

EXERCISE 66.

Perform the operation indicated:

1. $(x^2)^4$.
2. $(x^4)^2$.
3. $(xy^3)^5$.
4. $(-x^2)^3$.
5. $(-xy^3)^4$.
6. $(a^2b^5)^{10}$.
7. $(-ab^2c^3)^6$.
8. $(c^3b^6)^3$.
9. $\left(\frac{a}{2}\right)^3$.
10. $\left(\frac{2x}{4}\right)^2$.

11. $\left(\frac{xy}{b}\right)^4$.

12. $\left(\frac{x^2y^3}{2a}\right)^5$.

13. $\left(\frac{3abc^3}{4xy^2}\right)^4$.

14. $\left(-\frac{ab^2}{2x}\right)^3$.

15. $\left(\frac{-7am^2}{8bn^2}\right)^3$.

16. $\left(-\frac{5xy^2}{mn^2}\right)^5$.

17. $(a+b)^2$.

18. $(x+y)^3$.

19. $(a-2)^5$.

20. $(a+2b^2)^3$.

21. $(1-2a^2)^4$.

22. $(2x-3y^2)^3$.

23. $(3m-1)^7$.

24. $(ab-bc)^3$.

25. $(a+b+c)^3$.

26. $(a+b-1)^4$.

27. $(x+x^2+x^3)^3$.

28. $(1-x-2x^2)^4$.

29. $(3a-2b+4c)^3$.

30. $(a+2b-c+2d)^3$.

31. $(c+x-d+y)^4$.

32. $\left(\frac{x^a}{2}-\frac{y^{2b}}{3}\right)^3$.

33. $\left(\frac{a^x}{b^m}+\frac{b^m}{c^q}-\frac{c^p}{d^n}\right)^2$.

199. Evolution. The operation of finding a root of any expression is called **Evolution.**

Evolution is simply the process of finding one of the equal factors of an expression.

The symbols $\sqrt{\ }$, $\sqrt[3]{\ }$, $\sqrt[4]{\ }$ denote that the square, the cube, and the fourth root, respectively, are to be extracted.

To extract the root of a monomial.

By § 194, $(a^2)^5 = a^{10}$.

$\therefore \sqrt[5]{a^{10}} = a^2$.

Now if, in the given example, we substitute m for 2, and n for 5, we have, $\sqrt[n]{a^{mn}} = a^m$. This example is perfectly general.

200. Hence, to extract the root of any monomial expression, *divide the exponent of each factor by the index of the root, and take the product of the factors thus obtained.*

Thus, $\sqrt[3]{x^{12}} = x^4$; $\sqrt[3]{125\,y^6} = \sqrt[3]{5^3y^6} = 5\,y^2$;

$$\sqrt[n]{x^{2n}y^{3mn}} = x^2y^{3m}.$$

From rules already set forth, it is evident:

I. *That an even root of a positive number may be plus or minus.*

II. *That an odd root will be plus or minus, as the expression is plus or minus.*

III. *That a negative expression can have no even root.*

EXERCISE 67.

Perform the operation indicated:

1. $\sqrt{a^6}$.
2. $\sqrt{b^8}$.
3. $\sqrt[3]{x^9y^3}$.
4. $\sqrt{m^2n^4}$.
5. $\sqrt{x^2y^8z^{16}}$.
6. $\sqrt[5]{32\,a^{10}}$.
7. $\sqrt{4\,a^2}+\sqrt{9\,x^6}$.
8. $\sqrt[3]{-125\,a^3}+\sqrt[4]{256\,x^4}$.
9. $\sqrt{\dfrac{x^2}{y^4}}$.
10. $\sqrt{\dfrac{x^{2c}\,y^{4a}}{m^{8c}}}$.
11. $\sqrt[3]{\dfrac{a^{3x}}{b^{9y}}}+\sqrt[3]{\dfrac{b^{3m}}{a^{9n}}}$.
12. $\sqrt[4]{\dfrac{16\,x^{4m}}{81\,b^{4n}}}+\sqrt[3]{\dfrac{125\,x^{3x}}{343\,y^{6x}}}+\sqrt[3]{\dfrac{8\,x^{3m}\,y^{12a}}{27\,m^{3n}\,n^{6b}}}$.

To extract the square root of a polynomial.

By multiplication we find that $a^2+2\,ab+b^2$ is the square of $a+b$; conversely, $a+b$ must be the square root of $a^2+2\,ab+b^2$.

Performing the operation, we have,

$$\begin{array}{r|l}
\multicolumn{1}{r}{a^2+2\,ab+b^2} & \underline{\,a+b\,} \\
\multicolumn{1}{r}{\underline{a^2\qquad\qquad\quad}} & \\
2\,a+b & 2\,ab+b^2 \\
 & \underline{2\,ab+b^2}
\end{array}$$

In the above, a is the root of a^2. Taking a, therefore, for the first term of the root, and squaring it, we have a^2, which, subtracted from $a^2 + 2\,ab + b^2$, leaves $2\,ab + b^2$. By doubling the root already found, and dividing it into $2\,ab$, we obtain b, the second term of the root. Adding b, the second term of the root, to $2\,a$, the trial divisor, we obtain $2\,a + b$, which, multiplied by b, gives $2\,ab + b^2$.

201. Hence, to find the square root of a polynomial:

I. *Arrange the terms according to the powers of some letter.*

II. *Take the square root of the first term for the first term of the root.*

III. *Subtract the square of the root thus found from the given expression.*

IV. *Divide the first term of the resulting dividend by twice the root already found.*

V. *Place the quotient thus found for the second term of the root, and annex it to the trial divisor for a complete divisor.*

VI. *Multiply the completed divisor by the last term of the root, subtract the product from the dividend, and proceed with the remainder, if any, as before.*

Example (1): Extract the square root of

$$a^4 - a^3b + \frac{3\,a^2b^2}{4} - \frac{ab^3}{4} + \frac{b^4}{16}.$$

$$\begin{array}{r|l|l}
 & a^4 - a^3b + \frac{3\,a^2b^2}{4} - \frac{ab^3}{4} + \frac{b^4}{16} & a^2 - \frac{ab}{2} + \frac{b^2}{4} \\
 & a^4 & \\
\hline
2\,a^2 - \frac{ab}{2} & \quad - a^3b + \frac{3\,a^2b^2}{4} - \frac{ab^3}{4} + \frac{b^4}{16} & \\
 & \quad - a^3b + \frac{a^2b^2}{4} & \\
\hline
2\,a^2 - ab + \frac{b^2}{4} & \qquad \frac{2\,a^2b^2}{4} - \frac{ab^3}{4} + \frac{b^4}{16} & \\
 & \qquad \frac{2\,a^2b^2}{4} - \frac{ab^3}{4} + \frac{b^4}{16} & \\
\hline
\end{array}$$

EXERCISE 68.

Extract the square root of the following:

1. $a^2 + 2ab + b^2$.
2. $a^4 - 4a^3 + 6a^2 - 4a + 1$.
3. $x^6 - 6x^5y + 15x^4y^2 - 20x^3y^3 + 15x^2y^4 - 6xy^5 + y^6$.
4. $9 + 37x^2 + 4x^4 - 30x - 20x^3$.
5. $m^4 + 4p^2 + 9q^2 + 4m^2p + 6m^2q + 12pq$.
6. $a^2 + 4b^2 + 9c^2 - 4ab - 6ac + 12bc$.
7. $4x^2 + 16xy + 16y^2 - 12xz + 9z^2 - 24yz$.
8. $4x^2y^2 - 20xy^2z + 28x^2yz + 25y^2z^2 - 70xyz^2 + 49x^2z^2$.
9. $\frac{x^2}{4} + x + 1 + y + \frac{y^2}{4} + \frac{xy}{2}$.
10. $\frac{4a^2x^2}{9} - abxy + \frac{9b^2y^2}{16}$.
11. $\frac{4x^2}{9y^2} - \frac{x}{z} - \frac{16x^2}{15yz} + \frac{9y^2}{16z^2} + \frac{6xy}{5z^2} + \frac{16x^2}{25z^2}$.
12. $x^6 + \frac{1}{x^6} - 6x^4 - \frac{6}{x^4} - 20 + 15x^2 + \frac{15}{x^2}$.

EXERCISE 69.

Extract the square root of the following:

1. $a^{2m} + 2a^m + 1$.
2. $a^{\frac{2a}{10b}} + 2a^{\frac{a}{5b}} + 1$.
3. $9a^{2m} + 24a^{m+p} + 16a^{2p}$.
4. $\frac{9x^{-4n}y^{6m+8}}{49z^{-10n-4}} - \frac{x^{-1}y^{-1}}{z^{-1}} + \frac{49x^{4n-2}y^{-10-6m}}{36z^{10n+2}}$.
5. $m^{\frac{8}{9}} - 2n^{-\frac{1}{63}} + m^{-\frac{6}{7}}$.
6. $x^{\frac{4}{7}} + 2x^{\frac{24}{35}} + x^{\frac{4}{5}}$.
7. $9x^{2m+2} + 42x^{4m-2} + 103x^{6m-6} + 126x^{8m-10} + 81x^{10m-14}$.

202. Square Root of Numbers. In formulating a rule for the extraction of the square root of numbers, it is necessary to find the relation which exists between the number of places in the given number and the number of places in the root.

Since

$$\begin{array}{rl} 1^2 = 1, & 1^2 = 1, \\ 9^2 = 81, & 10^2 = 100, \\ 99^2 = 9801, & 100^2 = 10000, \\ 999^2 = 998001, & 1000^2 = 1000000, \end{array}$$

it is evident that a root consisting of one place has one or two places in its square, and that for every additional place in the root two places are added to the square.

Therefore, if we point off a number into periods of two figures each (beginning at the right), *the number of periods will be the same as the number of places in the root.*

The last period to the left may consist of one or two figures.

Extract the square root of 7225.

$$\begin{array}{r|l} & 72'25 \;\underline{|\,85} \\ & 64 \\ \hline 165 & 8\,25 \\ & \underline{8\,25} \end{array}$$

Comparing this with the algebraic square $a^2 + 2ab + b^2$, we have, $7225 = 80^2 + 2(80 \times 5) + 5^2 = 6400 + 800 + 25$.

The root, 85, $= a + b = 80 + 5$, the figure 8 occupying the tens' position.

203. Hence, the binomial $a^2 + 2ab + b^2$ may be used as a formula in extracting the square root of numbers.

Extract the square root of 4489.

$$\begin{array}{r|l} & 44'89 \;\underline{|\,67} \\ a^2 = & 36\,00 \\ \hline 2a = 120 & 8\,89 \\ 2a + b = 127 & \underline{8\,89} \end{array}$$

If we keep in mind that *a* represents the portion of the root already found, and also represents tens with reference to the next figure of the root, the process will apply to all numbers.

204. *In pointing off periods in a number which has decimal places, we must begin at the decimal point, and if the right-hand period contains only one place, annex a cipher.*

Example: Extract the square root of 146.41.

$$\begin{array}{r|l}
 & 1'46.41 \;\underline{|\,12.1} \\
 & 1 \\
22 & 46 \\
 & \underline{44} \\
241 & 241 \\
 & \underline{241}
\end{array}$$

There will be as many decimal places in the root as there are decimal periods.

EXERCISE 70.

Extract the square root of the following:

1. 3600.
2. 25600.
3. 1369.
4. 409600.
5. 51336683776.
6. 780811249.
7. 3466383376.
8. .81.
9. .0121.
10. .0169.
11. .2209.
12. .00056644.
13. .000000000361.
14. $\frac{49}{81}$; $\frac{81}{169}$.
15. $\frac{1089}{8100}$; $\frac{1369}{4900}$.
16. $\frac{3083536}{3108169}$.
17. $97\frac{97}{99}$; $10\frac{8}{10}$.
18. $\frac{1}{1719}$; $\frac{10.300}{12.000}$.

To extract the cube root of a polynomial.

The cube of $a + b$ is $a^3 + 3a^2b + 3ab^2 + b^3$.

Conversely, $a + b$ is the cube root of $a^3 + 3a^2b + 3ab^2 + b^3$.

Performing the operations, we have,

$$\begin{array}{r|l}
\multicolumn{2}{r}{a^3 + 3a^2b + 3ab^2 + b^3 \;\underline{\big|\; a + b}} \\
\multicolumn{2}{r}{\underline{a^3 \qquad\qquad\qquad\qquad}} \\
3a^2 + 3ab + b^2 & 3a^2b + 3ab^2 + b^3 \\
 & \underline{3a^2b + 3ab^2 + b^3}
\end{array}$$

Examining the above, we find that the first term of the root is the cube root of the first term of the given expression a^3. If the cube of the first term of the root be subtracted from the given expression, there remains $3a^2b + 3ab^2 + b^3$. Dividing the first term of this remainder by three times the square of a, the first term of the root, we obtain the second term of the root, b.

Since $3a^2b + 3ab^2 + b^3 = b(3a^2 + 3ab + b^2)$, we find that we can complete the divisor by adding $3ab + b^2$ to the *trial divisor*, $3a^2$.

205. Hence to find the cube root of a polynomial:

I. *Arrange the expression according to the powers of some letter.*

II. *Write the cube root of the first term for the first term of the root, and subtract the cube of the root thus found from the given expression.*

III. *Arrange the remainder for a dividend.*

IV. *Take three times the square of the root already found for a trial divisor. Divide this into the first term of the dividend and write the quotient as the second term of the root.*

V. *Add to the trial divisor three times the product of the first and second terms of the root, also the square of the second term, and the divisor will be complete.*

VI. *Multiply the complete divisor by the last term of the root, and subtract the product from the dividend.*

VII. *Regarding the root already found as a single term, proceed as before until there is no remainder.*

Extract the cube root of

$$x^6 - 12x^5 + 54x^4 - 112x^3 + 108x^2 - 48x + 8.$$

$$\begin{array}{l|l}
 & \underline{\lfloor x^2 - 4x + 2} \\
 & x^6 - 12x^5 + 54x^4 - 112x^3 + 108x^2 - 48x + 8 \\
 & x^6 \\
\hline
3 \times \text{sq. of } x^2 = \quad 3x^4 & -12x^5 + 54x^4 - 112x^3 \\
3x^2(-4x) = -12x^3 & \\
\text{sq. of } 4x = \quad 16x^2 & \\
\hline
\text{Hence, divisor,} & \\
3x^4 - 12x^3 + 16x^2 & -12x^5 + 48x^4 - \quad 64x^3 \\
\hline
3(x^2 - 4x)^2 = 3x^4 - 24x^3 + 48x^2 & 6x^4 - \quad 48x^3 + 108x^2 - 48x + 8 \\
3\{(x^2 - 4x)(2)\} = \quad 6x^2 - 24x & \\
\text{sq. of } 2 = \qquad 4 & \\
\hline
\text{Hence, divisor,} & \\
3x^4 - 24x^3 + 54x^2 - 24x + 4 & 6x^4 - \quad 48x^3 + 108x^2 - 48x + 8 \\
\hline
\end{array}$$

EXERCISE 71.

Extract the cube root of the following:

1. $a^3 + 3a^2b + 3ab^2 + b^3$.
2. $x^6 - 6x^5y + 15x^4y^2 - 60x^3y^3 + 15x^2y^4 - 6xy^5 + y^6$.
3. $8x^3 - 84x^2y + 294xy^2 - 343y^3$.
4. $a^6 - 3a^5 + 3a^4 + 2a^3 - 6a^2 + 3a + 3 - \frac{3}{a} + \frac{1}{a^3}$.
5. $x^3 - 3x^2y + 3xy^2 - y^3 + 3x^2z - 6xyz + 3y^2z + 3xz^2 - 3yz^2 + z^3$.
6. $\frac{8}{27}x^3 - \frac{16}{15}x^2b + \frac{32}{25}xb^2 - \frac{64}{125}b^3$.
7. $\frac{a^3b^3}{8c^9}x^6 - \frac{b}{2c^5}x^5 + \frac{2}{3a^2b^4c}x^4 - \frac{8c^3}{27a^6b^9}x^3$.
8. $a^{\frac{9}{4}} - 3a^{\frac{13}{6}} + 3a^{\frac{25}{12}} - a^2$.
9. $x^{-6m+12} - 6x^{-7m+3} + 12x^{-8m-6} - 8x^{-9m-15}$.

CUBE ROOT OF NUMBERS.

206. In formulating a rule for extracting the cube root of numbers, the first step necessary is to find the relation which exists between the number of places in a cube and its root.

Since

$$
\begin{aligned}
1^3 &= 1, & 1^3 &= 1,\\
9^3 &= 729, & 10^3 &= 1000,\\
99^3 &= 970299, & 100^3 &= 1000000,\\
999^3 &= 997002999, & 1000^3 &= 1000000000,
\end{aligned}
$$

it is evident that the cube of a number of one place may have one, two, or three places, and that for each additional place in the root there are three additional places in the cube.

Therefore if a number be divided into periods of three figures each beginning at the right, there will be as many figures in the root as there are periods in the cube. The last period to the left may consist of one, two, or three figures.

207. If the number has decimal places, the pointing off should begin at the decimal point, and if there are not sufficient figures to complete the last right-hand period, ciphers should be annexed.

208. There will be three times as many decimal places in the cube as there are decimal places in the root.

If the given number is not a perfect cube, an approximate root may be found to any required number of decimal places by annexing ciphers.

Comparing a given cube and its root with the algebraic binomial cube and its root, we shall find a simple rule for the solution of the cube root of numbers.

If
$$a + b = 12,$$
$$a^3 + 3a^2b + 3ab^2 + b^3 = 1728.$$

Now, in the roots $a+b$ and 12, a and 1 may be regarded as occupying the tens' position, b and 2 the units' position. 12 may be written $10+2$.

Substituting these values for a and b in the cube $a^3+3a^2b+3ab^2+b^3$, we have the result, $1000+600+120+8=1728$.

Subtracting a^3 from the given cube, we have, $3a^2b+3ab^2+b^3$; and removing from this the second term of the root, we have, $3a^2+3ab+b^2$, which expression is the complete divisor.

209. Hence, *the complete divisor equals 3 times the square of the first figure of the root (considered as tens), plus 3 times the first (as tens) by the second figure of the root, plus the square of the second figure of the root.*

210. If the root consists of three or more figures, that part of the root already found should be treated as the first figure in applying the above rule.

Again, decomposing 120, we have,

$$100^3 = 1000000,$$

$$120^3 = 1728000.$$

That is, the cube of the first figure of the root is contained wholly in the first period of the cube; the cube of the first two figures in the first two periods, etc.

(1) Extract the cube root of 256047875.

Since there are three periods, there will be three figures in the root.

$$\begin{array}{rr|l}
 & & 256'047'875\ \underline{|\ 635} \\
\text{The greatest cube in } 256 = 6^3. & & \underline{216} \\
3\times 60^2 = & 10800 & 40\,047 \\
3(60\times 3) = & 540 & \\
3^2 = & \underline{9} & \\
\text{Complete divisor} = & 11349 & \underline{34\,047} \\
3\times 630^2 = & 1190700 & 6\,000\,875 \\
3(630\times 5) = & 9450 & \\
5^2 = & \underline{25} & \\
\text{Complete divisor} = & 1200175 & 6\,000\,875
\end{array}$$

(2) Extract the cube root of 7 to three places of decimals:

$$\begin{array}{rr|l|l}
 & & 7.000000000 & \underline{1.912} \\
\text{Greatest cube in 7 is } 1^3. & & 1 & \\ \cline{3-3}
3\times 10^2 = & 300 & 6\,000 & \\
3(10\times\ 9) = & 270 & & \\
9^2 = & \underline{\ 81} & & \\
\text{Complete divisor} = & 651 & 5\,859 & \\ \hline
3\times 190^2 = & 108300 & 141000 & \\
3(190\times 1) = & 570 & & \\
1^2 = & \underline{1} & & \\
\text{Complete divisor} = & 108871 & 108871 & \\ \hline
3\times 1910^2 = & 10944300 & 32129000 & \\
3\,(1910\times 2) = & 10460 & & \\
2^2 = & \underline{4} & & \\
\text{Complete divisor} = & 10954764 & 21909528 &
\end{array}$$

In like manner the cube root of fractions may be found by taking the cube root of both numerator and denominator.

EXERCISE 72.

Extract the cube root of the following:

1. 571787.
2. 144125083907.
3. 18970.074963.
4. 305.909539272.
5. .171467.
6. 3. to three places of decimals.
7. 7.164132 to four places of decimals.
8. $\frac{3}{5}$; $\frac{5}{7}$; $\frac{7}{11}$; $\frac{11}{13}$.

211. Since the fourth power is the square of the square, the sixth power the square of the cube, the eighth power the square of the fourth power, it follows that the fourth root may be found by extracting the square root twice, the sixth root by extracting the square root and the cube root, the eighth root by extracting the square root three times. In like manner any root may be extracted when the index of the power is a product of the factors 2 and 3.

EXERCISE 73.

Extract the fourth root of the following:

1. $x^4 + 4x^3 + 6x^2 + 4x + 1.$
2. $16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4.$
3. $\frac{1}{x^4} - \frac{8}{x^3}y + \frac{24}{x^2}y^2 - \frac{32}{x}y^3 + 16y^4.$

Extract the sixth root of the following:

4. $64x^6 - 192x^5 + 240x^4 - 160x^3 + 60x^2 - 12x + 1.$
5. $729x^6 - 729x^5y + \frac{1215}{4}x^4y^2 - \frac{135}{2}x^3y^3 + \frac{135}{16}x^2y^4 - \frac{9}{16}xy^5 + \frac{y^6}{64}.$
6. $\frac{a^{12}}{b^6} - 6\frac{a^{10}c^2}{b^5d} + 15\frac{a^8c^4}{b^4d^2} - 20\frac{a^6c^6}{b^3d^3} + 15\frac{a^4c^8}{b^2d^4} - 6\frac{a^2c^{10}}{bd^5} + \frac{c^{12}}{d^6}.$

Extract the eighth root of the following:

7. $x^{16} - 8x^{14} + 28x^{12} - 56x^{10} + 70x^8 - 56x^6 + 28x^4 - 8x^2 + 1.$
8. $256a^8 - \frac{1024a^7x}{3y} + \frac{1792a^6x^2}{9y^2} - \frac{1792a^5x^3}{27y^3} + \frac{1120a^4x^4}{81y^4} - \frac{448a^3x^5}{243y^5} + \frac{112a^2x^6}{729y^6} - \frac{16ax^7}{2187y^7} + \frac{x^8}{6561y^8}.$

CHAPTER XVI.

SIMPLE INDETERMINATE EQUATIONS.

212. If a single equation is given, involving two unknown quantities, with no condition imposed, the quantities may have an indefinite number of values. Such an equation is called an **Indeterminate Equation.**

EXAMPLE: $9x - 11y = 8$.

$$\text{In this equation, if } \begin{cases} x = 0, & y = -\frac{8}{11}; \\ x = 1, & y = \frac{1}{11}; \\ x = 2, & y = \frac{10}{11}; \\ x = 3, & y = \frac{19}{11}; \\ x = 4 & y = \frac{28}{11}. \end{cases}$$

$$\text{if } \begin{cases} y = 0, & x = \frac{8}{9}; \\ y = 1, & x = \frac{10}{9}; \\ y = 2, & x = \frac{10}{3}; \\ y = 3, & x = \frac{41}{9}; \\ \text{and so on.} \end{cases}$$

213. If, however, on such an equation certain conditions be imposed, such that the value of the unknown quantities be *positive*, or be *positive integers*, the solutions may be limited.

214. Every equation of the first degree may be reduced to the form $ax \pm by = \pm c$.

Here a, b, and c are integers and have no common factor; for if they have a common factor, the equation may be reduced still further by division.

215. If in the above equation a and b have a common factor which does not divide c, the solution for positive integers is impossible. For if we divide both members of the equations by the common factor of a and b, we shall have the left-hand member integral and the right-hand member fractional, which is impossible.

Example (1): Solve for positive integral values the equation

$$8x + 65y = 81.$$

Transposing, $$8x = 81 - 65y.$$

$$\therefore\ x = 10 - 8y + \frac{1-y}{8}.$$

$$\therefore\ x + 8y - 10 = \frac{1-y}{8}.$$

Now, since x and y must be integral, $x + 8y - 10$ will be integral, and as a result $\frac{1-y}{8}$ will be integral.

Let m, an integer, equal $\frac{1-y}{8}$.

Then $$\frac{1-y}{8} = m,$$

and $$1 - y = 8m.$$

$$\therefore\ y = 1 - 8m.$$

Substituting this value of y in the original equation, we have,

$$8x + 65 - 520m = 81,$$

$$8x = 16 + 520m,$$

$$x = 2 + 65m.$$

Now, if m equal 0 or any positive integer, x will be a positive integer, but m cannot have a negative value if x remain positive.

In the equation $y = 1 - 8m$, if m equal 0 or any negative integer, y will be positive, but m cannot have a positive value if y remain positive.

Tabulating the results for values of m, this will readily be seen.

If $m = 0$, $x = 2$, $y = 1$,
$m = +1$, $x = 67$, $y = -7$,
$m = -1$, $x = -63$, $y = +9$.

Therefore the only positive integral values of x and y which satisfy the equation are 2 and 1, respectively.

Example (2): $5x - 16y = 24$.

Transposing, $5x = 24 + 16y$.

$$\therefore\ x = 4 + 3y + \frac{4+y}{5}$$

$$\therefore\ x - 3y - 4 = \frac{4+y}{5}.$$

Let $\frac{4+y}{5} = m$.

Then $4 + y = 5m$,

$$y = 5m - 4.$$

Substituting this value of y in the original equation, we have,

$$5x - 80m = -40,$$

$$x = 16m - 8.$$

If $m = 1$, or any other positive integer, x will be positive.

In the equation $y = 5m - 4$, y will be positive for all integral values greater than 0.

Tabulating the results, we have:

If $m = 1$, $x = 8$, $y = 1$,
$m = 2$, $x = 24$, $y = 6$,
$m = 3$, $x = 40$, $y = 11$.

Thus, we see, in the given equation, x and y may have any number of positive integral values.

Example (3): Solve $5x + 7y = 33$ for the least positive integral values of x and y.

$$5x = 33 - 7y,$$

$$x = 6 - y + \frac{3 - 2y}{5}. \qquad (1)$$

Let $$\frac{3 - 2y}{5} = m.$$

Then $$3 - 2y = 5m,$$

$$2y = 3 - 5m,$$

$$y = 1 - 2m + \frac{1 - m}{2}. \qquad (2)$$

Let $$\frac{1 - m}{2} = p.$$

Then $$1 - m = 2p,$$

$$m = 1 - 2p.$$

$\therefore$ since m is an integer, $1 - 2p$ must be an integer.
Substituting this value of m in (2), we have,

$$y = 5p - 1. \qquad (3)$$

Again, substituting in (1), we have,

$$x = 8 - 7p.$$

Now, if $p = 1$, then will $y = 4$, and $x = 1$, which are the least positive integral values for x and y.

Example (4): Find the least number that, when divided by 9, will give a remainder of 6, and when divided by 7 will give a remainder of 1.

Let x and y be the quotients, respectively, and n the number.

Then, $$\frac{n - 6}{9} = x; \quad \frac{n - 1}{7} = y,$$

and $$n = 9x + 6; \; n = 7y + 1.$$

$$\therefore\ 9x + 6 = 7y + 1.$$

$$\therefore\ 9x - 7y = -5.$$

Solving,
$$9x = 7y - 5,$$
$$x = \frac{7y-5}{9}. \tag{1}$$
$$\frac{7y-5}{9} = m,$$
$$7y - 5 = 9m,$$
$$7y = 9m + 5,$$
$$y = m + \frac{2m+5}{7}. \tag{2}$$
$$\frac{2m+5}{7} = p,$$
$$2m + 5 = 7p,$$
$$2m = 7p - 5,$$
$$m = 3p - 2 + \frac{p-1}{2}. \tag{3}$$
$$\frac{p-1}{2} = q,$$
$$p - 1 = 2q,$$
$$p = 2q + 1.$$

Substituting this value in (3), (2), and (1), respectively, we find, $x = 1$ and $y = 2$.

$$\therefore\ n = 9x + 6 = 7y + 1 = 15.$$

EXERCISE 74.

Solve the following in positive integers:

1. $2x + 5y = 23.$
2. $5x + 3y = 58.$
3. $3x + 4y = 39.$
4. $x + 3y = 47.$
5. $19x + 17y = 53.$
6. $4x + 9y = 89.$
7. $8x + 17y = 67.$
8. $13x + y = 40.$
9. $16x + 15y = 78.$
10. $7x + 10y = 412.$

Solve the following in least positive integers:

11. $11x - 9y = 24.$	**16.** $29x - 2y = 54.$
12. $7x - 3y = 32.$	**17.** $17x - 3y = 76.$
13. $8x - 17y = 23.$	**18.** $520x - 301y = 1779.$
14. $3x - 9y = 51.$	**19.** $3x - 69y = 231.$
15. $x - 13y = 67.$	**20.** $4x - 5y = 1.$

21. In how many different ways can the sum of $12.50 be paid with one-dollar pieces and fifty-cent pieces?

22. In how many ways can $50 be paid in five-dollar and two-dollar notes.

23. Find two fractions whose denominators are 3 and 5, and whose sum is $6\frac{8}{15}$.

24. Find two fractions whose denominators are $3\frac{1}{5}$ and $4\frac{1}{7}$, and whose sum is $11\frac{3}{464}$.

25. In how many ways can $7.10 be divided into quarter-dollars and dimes?

26. Divide 400 into two such parts, so that if one be divided by 16, and the other by 19, the respective remainders shall be 8 and 10, and the sum of the quotients shall be 22.

27. How many fractions are there with denominators of 24 and 36 whose sum is $\frac{50}{72}$?

CHAPTER XVII.

THEORY OF EXPONENTS.

Fractional Exponents.

216. According to the laws already shown:

(*a*) $a^m = a$ taken m factors times.

(*b*) $a^m \times a^n = a^{m+n}$.

(*c*) $(a^m)^n = a^{mn}$.

(*d*) $\frac{a^m}{a^n} = a^{m-n} = a^{-n+m} = a^{-(n-m)} = \frac{1}{a^{n-m}}$.

In like manner,

(*e*) $\sqrt[n]{a^{mn}} = a^m$.

(*f*) $(ab)^m = a^m b^m$.

(*g*) $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$.

217. The above laws hold true whatever the form of the exponent, positive or negative, integral or fractional. If, then, $a^m \times a^m = a^{2m}$ is a fundamental law, it follows that $a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{2}{2}} = a$. What, then, is the meaning of $a^{\frac{1}{2}}$? From the result shown, $a^{\frac{1}{2}}$ must be a number that, when squared, equals a, and a must be a number whose square root equals $a^{\frac{1}{2}}$. That is, $a^{\frac{1}{2}} = \sqrt{a}$.

Again, $a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a^{\frac{3}{3}} = a$.

Hence, $a^{\frac{1}{3}} = \sqrt[3]{a}$.

By like reasoning, $a^{\frac{2}{3}} \times a^{\frac{2}{3}} \times a^{\frac{2}{3}} = a^{\frac{6}{3}} = a^2$.

That is, $a^{\frac{2}{3}}$ is one of three equal factors which produce a^2.
$\therefore\ a^{\frac{2}{3}} = \sqrt[3]{a^2}$.

In like manner, $a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times a^{\frac{1}{n}} \cdots$ to n factors equals

$$a^{\frac{1}{n}+\frac{1}{n}+\frac{1}{n}\text{ to } n \text{ times}} = a^{\frac{n}{n}} = a.$$

That is, $a^{\frac{1}{n}}$ is one of n equal factors which produce a.

Hence, $$a^{\frac{1}{n}} = \sqrt[n]{a}.$$

In like manner, the meaning of $a^{\frac{m}{n}}$ is made clear.

For $a^{\frac{m}{n}} \times a^{\frac{m}{n}} \times a^{\frac{m}{n}} \cdots$ to n factors times equals $a^{\frac{nm}{n}} = a^m$.

That is, $a^{\frac{m}{n}}$ is one of n equal factors which produce a^m.

Hence, $$a^{\frac{m}{n}} = \sqrt[n]{a^m}.$$

218. From this last, the rule may be deduced *that, in a fractional exponent, the numerator denotes a power and the denominator a root.*

Negative Exponents.

In sections (71, 72), we proved the theorems $a^0 = 1$, and $a^{-1} = \frac{1}{a}$. It remains only to extend this proof further to negative exponents.

From the given law, we have

$$a^m \times a^{-n} = a^{m-n};\ \text{but } \frac{a^m}{a^n} = a^{m-n}.$$

$$\therefore\ a^m \times a^{-n} = \frac{a^m}{a^n}.\quad \therefore\ a^{-n} = \frac{1}{a^n}.$$

219. That is (73), *any quantity with a negative exponent is equal to the reciprocal of that quantity with an equal positive exponent.*

220. *The reciprocal of a quantity is* 1 *divided by that quantity. The reciprocal of a fraction is that fraction inverted.*

Hence a^{-n} is the reciprocal of a^n; that is, $a^{-n} = \frac{1}{a^n}$; $a^n = \frac{1}{a^{\ n}}$; $\therefore\ a^{-n} \times a^n = 1$.

If in the quantity a^{m-n} $m > n$, the result will be a positive exponent; but if $n > m$, the result will be otherwise, and we shall have the expression $a^{-(n-m)} = a^{m-n}$.

For, $\quad a^m \div a^n = \frac{a^m}{a^n} = \frac{1}{a^{n-m}} = a^{-(n-m)} = a^{m-n}$.

From what has been shown:

$$a^{\frac{2}{3}} = \sqrt[3]{a^2},$$

$$a^{\frac{3}{2}} = \sqrt{a^3},$$

$$a^{\frac{6}{3}} = \sqrt[3]{a^6} = a^2,$$

$$a^{-2} = \frac{1}{a^2},$$

$$a^{-n} = \frac{1}{a^n},$$

$$a^{-\frac{1}{n}} = \frac{1}{a^{\frac{1}{n}}} = \frac{1}{\sqrt[n]{a}},$$

it will appear that quantities with fractional and negative exponents are simply another form of notation for quantities which we already had the means of representing.

To show that $(ab)^{\frac{1}{n}} = a^{\frac{1}{n}}b^{\frac{1}{n}}$, **and** $(ab)^{-\frac{1}{n}} = \frac{1}{a^{\frac{1}{n}}b^{\frac{1}{n}}}$:

For, $\quad (ab)^n = a^n b^n$.

Substituting $\frac{1}{n}$ for n, we have,

$$(ab)^{\frac{1}{n}} = a^{\frac{1}{n}}b^{\frac{1}{n}},$$

and

$$(ab)^{-\frac{1}{n}} = \frac{1}{(ab)^{\frac{1}{n}}} = \frac{1}{a^{\frac{1}{n}}b^{\frac{1}{n}}}.$$

In like manner,

$$\left(\frac{a}{b}\right)^{\frac{1}{n}}=\frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}, \text{ and } \left(\frac{a}{b}\right)^{-\frac{1}{n}}=\frac{b^{\frac{1}{n}}}{a^{\frac{1}{n}}}.$$

For, $$\left(\frac{a}{b}\right)^{n}=\frac{a^n}{b^n}.$$

Substituting $\frac{1}{n}$ for n, we have,

$$\left(\frac{a}{b}\right)^{\frac{1}{n}}=\frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}.$$

Again, $$\left(\frac{a}{b}\right)^{-\frac{1}{n}}=\frac{a^{-\frac{1}{n}}}{b^{-\frac{1}{n}}}=\frac{b^{\frac{1}{n}}}{a^{\frac{1}{n}}}.$$

In like manner, it may be shown that

$$(a^{\frac{1}{m}})^{\frac{1}{n}}=a^{\frac{1}{mn}},$$

and $$(a^{-\frac{1}{n}})^{\frac{1}{n}}=a^{-\frac{1}{mn}}=\frac{1}{a^{\frac{1}{mn}}},$$

and $$(a^{\frac{1}{m}})^{-\frac{1}{n}}=a^{-\frac{1}{mn}}=\frac{1}{a^{\frac{1}{mn}}},$$

and $$(a^{-\frac{1}{m}})^{-\frac{1}{n}}=a^{\frac{1}{mn}}.$$

To show that $$a^{\frac{m}{n}}=a^{\frac{mp}{np}}.$$

Let $$x=a^{\frac{m}{n}}.$$

Raising both members of the equation to the npth power, we have,

$$x^{np}=a^{mp}.$$

Taking the npth root of both members, we have $x=a^{\frac{mp}{np}}=a^{\frac{m}{n}}$.

EXERCISE 75.

1. Simplify $(x^{\frac{3}{5}}y^{\frac{3}{7}})^{\frac{3}{4}}$.

2. Multiply together $a^{-\frac{1}{2}}$, $b^{-\frac{1}{2}}$, $a^{\frac{3}{2}}$, $b^{\frac{5}{2}}$, and c^2.

3. Simplify $\left(\frac{a^2y}{b}\right)^{\frac{1}{3}}$, $\left(\frac{b^3x^{\frac{1}{3}}}{c^{\frac{2}{3}}}\right)^{\frac{1}{5}}$.

4. Express as radicals $a^{\frac{2}{3}}$, $b^{\frac{4}{5}}$, $c^{1\frac{1}{2}}$, $d^{3\frac{6}{11}}$, $x^{\frac{a}{b}}$, $b^{\frac{3m}{4p}}$.

5. Express with fractional exponents:

$$\sqrt[3]{a^2},\ \sqrt[4]{b^{11}c^{13}},\ \sqrt[9]{x^5y^7},\ \sqrt[n]{x^ay^b}.$$

6. Multiply $a^{\frac{1}{2}} - b^{\frac{1}{2}}$ by $a^{\frac{1}{2}} + b^{\frac{1}{2}}$.

7. Divide $a - b$ by $a^{\frac{1}{3}} - b^{\frac{1}{3}}$.

8. Multiply $x^{\frac{1}{2}} - x^{\frac{1}{4}}y^{\frac{1}{4}} + y^{\frac{1}{2}}$ by $x^{\frac{1}{2}} + x^{\frac{1}{4}}y^{\frac{1}{4}} + y^{\frac{1}{2}}$.

9. Multiply $a^{\frac{1}{5}} - b^{\frac{1}{5}}$ by $a^{\frac{1}{5}} + b^{\frac{1}{5}}$.

10. Divide $a^{\frac{2}{n}} - 2a^{\frac{1}{n}} + 1$ by $a^{\frac{1}{n}} - 1$.

11. Multiply $a^{-\frac{1}{3}} + b^{-\frac{1}{3}}$ by $a^{-\frac{2}{3}} - a^{-\frac{1}{3}}b^{-\frac{1}{3}} + b^{-\frac{2}{3}}$.

12. Simplify $\left\{\frac{(a^m)^{\frac{1}{n}}(b^n)^{\frac{1}{m}}}{(c^{\frac{1}{n}})^{\frac{1}{m}}(d^{\frac{1}{m}})^{\frac{1}{n}}}\right\}^{pq}$.

13. Extract the cube root:

$$x^{\frac{3}{5}} - 3x^{\frac{2}{5}} + 3x^{\frac{1}{5}} - 1.$$

14. Write the square:

$$a^{\frac{3}{5}} - 1,\ a^{\frac{3}{7}} + b^{\frac{1}{5}},\ x^{\frac{3}{4}} + y^{\frac{3}{4}} + z^{\frac{3}{4}}.$$

15. Write the third power:

$$x^{-1} - y^{-3}.$$

16. Extract the fourth root:

$$x^{-\frac{4}{7}} - 4x^{-\frac{3}{7}}y^{\frac{1}{5}} + 6x^{-\frac{2}{7}}y^{\frac{2}{5}} - 4x^{-\frac{1}{7}}y^{\frac{3}{5}} + y^{\frac{4}{5}}.$$

CHAPTER XVIII.

RADICAL EXPRESSIONS.

221. All expressions affected with the radical sign are called **Radicals.**

222. Those expressions which are affected with the radical sign and of which the exact root cannot be obtained are called **Surds** or **Irrational Quantities.**

Thus $\sqrt[5]{a^4}$ or $a^{\frac{4}{5}}$ is called a surd, but $\sqrt[3]{a^9}$ or a^3 is not a surd, as the exact root can be taken.

Surds are similar which have the same surd factor.

223. A **mixed surd** is one which contains a rational and a surd factor.

Thus $5\sqrt{2}$ and $b\sqrt{ac}$ are mixed surds. The rational factors of mixed surds are regarded as the coefficients of the surds.

An **Entire Surd** is one which has no coefficient.

Thus $\sqrt{10}$, $\sqrt{ab^2}$, $\sqrt[3]{(a+b)^4}$.

I. REDUCTION OF RADICALS.

To reduce to a lower index.

EXAMPLE: (1) Reduce $\sqrt[4]{4}$.

$$\sqrt[4]{4} = \sqrt[4]{2^2} = 2^{\frac{2}{4}} = 2^{\frac{1}{2}} = \sqrt{2}.$$

(2) Reduce $\sqrt[6]{a^3}$.

$$\sqrt[6]{a^3} = a^{\frac{3}{6}} = a^{\frac{1}{2}} = \sqrt{a}.$$

EXERCISE 76.

Reduce the following:

1. $\sqrt[8]{100}$.
3. $\sqrt[6]{27}$.
5. $\sqrt[9]{343}$.
7. $\sqrt[15]{32\,a^5b^{10}}$.
2. $\sqrt[4]{9}$.
4. $\sqrt[10]{32}$.
6. $\sqrt[12]{64\,a^6}$.
8. $\sqrt[4]{100\,a^2b^2}$.
9. $\sqrt[4]{\dfrac{36\,x^2}{81\,a^2b^2}}$.
10. $\sqrt[6]{\dfrac{27\,b^3x^3}{125\,a^3y^3}}$.
11. $\sqrt[4]{\dfrac{(a+b+c)^2}{(a-b-c)^2}}$.

II. **To reduce a radical to a mixed surd. Also to reduce a mixed surd to an entire surd.**

Reduce to mixed surds:

EXAMPLES: (1) $\sqrt{125}$.

$$\begin{aligned}\sqrt{125} &= \sqrt{25 \times 5}\\ &= \sqrt{5^2 \times 5}\\ &= 5^{\frac{2}{2}} \times 5^{\frac{1}{2}}.\\ &= 5\sqrt{5}.\end{aligned}$$

(2) Reduce $\sqrt[3]{64\,a^4b^5}$.

$$\begin{aligned}\sqrt[3]{64\,a^4b^5} &= \sqrt[3]{4^3 \times a^3 \times b^3 \times a \times b^2}\\ &= 4^{\frac{3}{3}} \times a^{\frac{3}{3}} \times b^{\frac{3}{3}} \times a^{\frac{1}{3}} \times b^{\frac{2}{3}}\\ &= 4\,ab\sqrt[3]{ab^2}.\end{aligned}$$

(3) Reduce $\sqrt{\dfrac{(a-b)^3}{(a+b)^2}}$.

$$\begin{aligned}\sqrt{\frac{(a-b)^3}{(a+b)^2}} &= \frac{(a-b)^{\frac{3}{2}}}{(a+b)^{\frac{2}{2}}}\\ &= \frac{(a-b)(a-b)^{\frac{1}{2}}}{a+b}\\ &= \frac{a-b}{a+b}\sqrt{a-b}.\end{aligned}$$

Reduce to entire surds:

(4) $a^2\sqrt{x^2y^3z^5}$.

$$a^2\sqrt{x^2y^3z^5} = a^{\frac{4}{2}}x^{\frac{2}{2}}y^{\frac{3}{2}}z^{\frac{5}{2}} = \sqrt{a^4x^2y^3z^5}.$$

(5) $a + b\sqrt{(a+b)^3(b+c)^5}$.

$$a + b\sqrt{(a+b)^3(b+c)^5} = (a+b)^{\frac{2}{2}}(a+b)^{\frac{3}{2}}(b+c)^{\frac{5}{2}}$$
$$= \sqrt{(a+b)^5(b+c)^5}.$$

(6) $\frac{x}{y}\sqrt[2n]{\frac{x^{2n}y^{3n}p^n}{a^nb^{3n}c^{5n}}}$.

$$\frac{x}{y}\sqrt[2n]{\frac{x^{2n}y^{3n}p^n}{a^nb^{3n}c^{5n}}} = \frac{x^{\frac{2n}{2n}}}{y^{\frac{2n}{2n}}}\left(\frac{x^{2n}y^{3n}p^n}{a^nb^{3n}c^{5n}}\right)^{\frac{1}{2n}}$$
$$= \sqrt[2n]{\frac{x^{4n}y^np^n}{a^nb^{3n}c^{5n}y^{2n}}}.$$

EXERCISE 77.

Reduce the following to mixed surds:

1. $\sqrt{a^2b}$.
2. $\sqrt{a^5b^7x^9}$.
3. $\sqrt[3]{64\,a^4}$.
4. $\sqrt[4]{32\,a^5b^7}$.
5. $\sqrt{125\,a^3}$.
6. $\sqrt[3]{10000\,a^5}$.
7. $\sqrt[5]{64\,x^7b^8}$.
8. $\sqrt{75\,x^3}$.
9. $\sqrt[3]{108\,x^5y^7}$.
10. $\sqrt[5]{729\,x^6}$.
11. $\sqrt{-27\,a^3}$.
12. $\sqrt{-64\,x^7}$.
13. $6\sqrt[3]{-16\,a^5x^7}$.
14. $3\sqrt[5]{-1029\,a^4}$.

15. $\sqrt{\dfrac{(a-b)^5}{x^4}}$.

16. $\sqrt[4]{\dfrac{(a+b+c)^5}{(a-b-c)^8}}$.

17. $\sqrt[\frac{2a}{x}]{\dfrac{8\,a^2b^3}{16\,x^4y^8}}$.

18. $\sqrt[\frac{1}{x}]{\dfrac{x^2(a-a^2-a^3)}{a^2-2\,ab+b^2}}$.

19. $\sqrt[3]{\dfrac{x^4-4\,x^3+6\,x^2-4\,x+1}{a^3-3\,a^2+3\,a+1}}$.

Reduce the following to entire surds.

20. $3\sqrt{x^2y}$.

21. $a^2\sqrt{x^3y^2z^5}$.

22. $5\,a\sqrt[3]{x^4y^5z^7}$.

23. $\dfrac{2}{x}\sqrt[4]{\tfrac{1}{32}\,x^5y^6}$.

24. $\dfrac{7}{4}\sqrt[5]{\dfrac{64\,x^2y^6}{243\,z^5}}$.

25. $\dfrac{(a+b)^2}{(x+y)^2}\sqrt[3]{(a+b)^3(x+y)^4}$.

26. $2\,c\sqrt[3]{\dfrac{ab^2c^3}{8\,z^6}}$.

27. $\dfrac{1}{a+b}\sqrt{x^3(a^2+2\,ab+b^2)}$.

28. $\dfrac{1}{c^2}\sqrt[a]{a^{2a}b^{3a}c^{4a}}$.

29. $\dfrac{1}{(x+y)(a+b)^2}\sqrt[x]{(x+y)^{2x}(a+b)^{3x}}$.

30. $a\sqrt[3m]{x^{3m}y^{3m}z^{5m}}$.

31. $\dfrac{1}{(a+b)^2(x+y)^4}\sqrt[2n]{(a+b)^{4n}(x+y)^{8n}x^{3n}}$.

III. **To reduce a fractional radical to a mixed surd in which the radical part shall be integral.**

EXAMPLES:

(1) Reduce $\sqrt{\frac{2}{3}}$.

$$\sqrt{\tfrac{2}{3}}=\sqrt{\tfrac{6}{9}}=\sqrt{6\times\tfrac{1}{9}}=\tfrac{1}{3}\sqrt{6}.$$

(2) Reduce $\sqrt[3]{\dfrac{5\,x}{3\,y^2}}$.

$$\sqrt[3]{\frac{5\,x}{3\,y^2}}=\sqrt[3]{\frac{45\,xy}{27\,y^3}}=\sqrt[3]{\frac{1}{27\,y^3}\times 45\,xy}=\frac{1}{3\,y}\sqrt[3]{45\,xy}.$$

(3) Reduce $\sqrt[3]{\dfrac{(x+y)(a-b)}{(a^2+2ab+b^2)(x^2-2xy+y^2)}}$.

$$\sqrt[3]{\frac{(x+y)(a-b)}{(a^2+2ab+b^2)(x^2-2xy+y^2)}}=\sqrt[3]{\frac{(x^2-y^2)(a^2-b^2)}{(a+b)^3(x-y)^3}}$$

$$=\sqrt[3]{(x^2-y^2)(a^2-b^2)\times\frac{1}{(a+b)^3(x-y)^3}}$$

$$=\frac{1}{(a+b)(x-y)}\sqrt[3]{(x^2-y^2)(a^2-b^2)}.$$

EXERCISE 78.

Reduce to a mixed surd with an integral radical part:

1. $\sqrt{\dfrac{1}{5}}$.
2. $\sqrt{\dfrac{1}{6}}$.
3. $\sqrt{\dfrac{a}{7}}$.
4. $\sqrt[3]{\dfrac{x^2}{2}}$.
5. $\sqrt[3]{\dfrac{3}{25}}$.
6. $\sqrt{\dfrac{4a^2}{5b}}$.
7. $\sqrt[3]{\dfrac{a^2b^2c^2}{xyz}}$.
8. $\sqrt{\dfrac{3}{2}a^2b}$.
9. $\dfrac{3}{2}\sqrt[3]{\dfrac{2}{4ab}}$.
10. $\dfrac{1}{2}\sqrt[4]{\dfrac{16a^2b^2}{18x^3y^3}}$.
11. $\dfrac{7}{16}\sqrt{\dfrac{16}{7}}$.
12. $\sqrt[3]{\dfrac{a+b}{a-b}}$.
13. $\sqrt[3]{\dfrac{a^2-2ab+b^2}{a^2+2ab+b^2}}$.
14. $\sqrt[3]{\dfrac{x^2-xy+y^2}{x^2+xy+y^2}}$.
15. $\dfrac{3}{15}\sqrt{\dfrac{a^{-2}b^{-2}c^{-3}}{(a+b)^{-1}(x+y)^{-1}}}$.

IV. To reduce radicals to a common index.

(1) Reduce $\sqrt[3]{2}$ and $\sqrt[5]{3}$ to a common index.

$$\sqrt[3]{2}=2^{\frac{1}{3}}=2^{\frac{5}{15}}=\sqrt[15]{2^5}=\sqrt[15]{32},$$

$$\sqrt[5]{3}=3^{\frac{1}{5}}=3^{\frac{3}{15}}=\sqrt[15]{3^3}=\sqrt[15]{27}.$$

$$\therefore\ \sqrt[3]{2}\text{ and }\sqrt[5]{3}=\sqrt[15]{32}\text{ and }\sqrt[15]{27}.$$

In the above example the question may arise, which is the greater, $\sqrt[3]{2}$ or $\sqrt[5]{3}$. The reduction shows $\sqrt[3]{2}$ to be the greater.

(2) Reduce $\sqrt[3]{5}$ and $\sqrt[4]{15}$ to a common index.

$$\sqrt[3]{5} = 5^{\frac{1}{3}} = 5^{\frac{4}{12}} = \sqrt[12]{5^4} = \sqrt[12]{625},$$

$$\sqrt[4]{15} = 15^{\frac{1}{4}} = 15^{\frac{3}{12}} = \sqrt[12]{15^3} = \sqrt[12]{3375}.$$

$$\therefore\ \sqrt[3]{5} \text{ and } \sqrt[4]{15} = \sqrt[12]{625} \text{ and } \sqrt[12]{3375}.$$

EXERCISE 79.

Reduce to a common index:

1. $\sqrt[3]{2}$ and $\sqrt[4]{3}$.
2. $\sqrt[4]{5}$ and $\sqrt[5]{6}$.
3. $\sqrt{3}$ and $\sqrt[3]{7}$.
4. $\sqrt{a}$ and $\sqrt[5]{x^2}$.
5. $\sqrt[4]{x^3}$ and $\sqrt[6]{ax^5}$.
6. $\sqrt[7]{ax^2}$ and $\sqrt[9]{x^2y}$.
7. $\sqrt[3]{a^2x^2y}$ and $\sqrt[11]{b^2y^2z^2}$.
8. $\sqrt{a+b}$ and $\sqrt[3]{a-b}$.
9. $\sqrt[3]{a^2-1}$ and $\sqrt[5]{a^2+1}$.
10. $\sqrt[3]{x^2+x-1}$ and $\sqrt[5]{x^3+x^2+x}$.

Arrange in order of magnitude:

11. $\sqrt{2}$, $\sqrt[3]{3}$, $\sqrt[4]{4}$.
12. $\sqrt{\frac{1}{2}}$, $\sqrt[3]{\frac{1}{3}}$, $\sqrt[4]{\frac{1}{5}}$.
13. $2\sqrt{\frac{1}{3}}$, $2\sqrt[3]{\frac{1}{2}}$, $2\sqrt[6]{\frac{1}{8}}$.
14. $\sqrt{19}$, $\sqrt[3]{100}$, $\sqrt[4]{300}$.

CHAPTER XIX.

ADDITION OF RADICALS.

224. It is evident from the laws of addition that $a^{\frac{1}{2}}$ and $a^{\frac{1}{3}}$ can be added together only by connecting them with the sign of addition.

Thus, the sum of $a^{\frac{1}{2}}$ and $a^{\frac{1}{3}} = a^{\frac{1}{2}} + a^{\frac{1}{3}}$; but $a^{\frac{1}{2}}$ may be added directly to $a^{\frac{1}{2}}$, giving a result $2\,a^{\frac{1}{2}}$.

225. *Two or more radicals, therefore, which are similar, may be added together by taking the sum of their coefficients and prefixing it to the common surd factor.*

Radicals are said to be *similar* if, when reduced to the simplest form, they have the same surd factor.

Dissimilar radicals may be added by connecting them with the sign plus.

EXAMPLES:

(1) Add together $\sqrt{8}$, $\sqrt{32}$, $\sqrt{162}$.

Now,
$$\begin{aligned}\sqrt{8} &= \sqrt{2^2 \times 2} = 2\sqrt{2},\\ \sqrt{32} &= \sqrt{4^2 \times 2} = 4\sqrt{2},\\ \sqrt{162} &= \sqrt{9^2 \times 2} = 9\sqrt{2}.\end{aligned}$$
$$\therefore\ \sqrt{8} + \sqrt{32} + \sqrt{162} = (2 + 4 + 9)\sqrt{2} = 15\sqrt{2}.$$

(2) Add together $12\sqrt[5]{64}$, $5\sqrt[5]{2048}$, $7\sqrt[5]{486}$.

Now,
$$\begin{aligned}12\sqrt[5]{64} &= 12\sqrt[5]{2^5 \times 2} = 24\sqrt[5]{2},\\ 5\sqrt[5]{2048} &= 5\sqrt[5]{4^5 \times 2} = 20\sqrt[5]{2},\\ 7\sqrt[5]{486} &= 7\sqrt[5]{3^5 \times 2} = 21\sqrt[5]{2}.\end{aligned}$$
$$\therefore\ 12\sqrt[5]{64} + 5\sqrt[5]{2048} + 7\sqrt[5]{486} = (24 + 20 + 21)\sqrt[5]{2} = 65\sqrt[5]{2}.$$

(3) Add together $\sqrt{4a^2b^3}$, $\sqrt[4]{(a+b)^4b^2}$, $\sqrt[6]{(a-b)^{12}b^3}$.

Now, $\sqrt{4a^2b^3} = \sqrt{(2ab)^2 \times b} = (2ab) \times b^{\frac{1}{2}} = 2ab\sqrt{b}$,

$$\sqrt[4]{(a+b)^4b^2} = (a+b)\,b^{\frac{2}{4}} = (a+b)\,b^{\frac{1}{2}} = (a+b)\sqrt{b},$$

$$\sqrt[6]{(a-b)^{12}b^3} = (a-b)^{\frac{12}{6}}b^{\frac{3}{6}} = (a-b)^2b^{\frac{1}{2}} = (a-b)^2\sqrt{b}.$$

$$\therefore\ \sqrt{4a^2b^3} + \sqrt[4]{(a+b)^4b^2} + \sqrt[6]{(a-b)^{12}b^3}$$
$$= (2ab + a + b + a^2 - 2ab + b^2)\sqrt{b}$$
$$= (a^2 + a + b^2 + b)\sqrt{b}.$$

(4) Add together $\sqrt{\frac{1}{2}}$, $2\sqrt{\frac{1}{8}}$, $5\sqrt{\frac{1}{18}}$.

Now $\sqrt{\frac{1}{2}} = \sqrt{\frac{2}{4}} = \sqrt{\frac{1}{4} \times 2} = \frac{1}{2}\sqrt{2}.$

$$2\sqrt{\tfrac{1}{8}} = 2\sqrt{\tfrac{2}{16}} = 2\sqrt{\tfrac{1}{16} \times 2} = \tfrac{1}{2}\sqrt{2}.$$

$$5\sqrt{\tfrac{1}{18}} = 5\sqrt{\tfrac{2}{36}} = 5\sqrt{\tfrac{1}{36} \times 2} = \tfrac{5}{6}\sqrt{2}.$$

$$\therefore\ \sqrt{\tfrac{1}{2}} + 2\sqrt{\tfrac{1}{8}} + 5\sqrt{\tfrac{1}{18}} = (\tfrac{1}{2} + \tfrac{1}{2} + \tfrac{5}{6})\sqrt{2} = 1\tfrac{5}{6}\sqrt{2}.$$

EXERCISE 80.

Add together the following:

1. $\sqrt{21}$, $\sqrt{9}$, $\sqrt{36}$.
2. $\sqrt{8}$, $\sqrt{18}$, $\sqrt{72}$.
3. $\sqrt{a^3}$, $\sqrt{c^2a}$, $\sqrt{ab^4}$.
4. $\sqrt{50}$, $\sqrt{200}$, $\sqrt{500}$.
5. $\sqrt[3]{24}$, $\sqrt[3]{81}$, $\sqrt[3]{375}$.
6. $\sqrt[3]{a^4b^6}$, $\sqrt[3]{ac^9}$, $\sqrt[3]{a^7d^9}$.
7. $\sqrt{49a^2}$, $\sqrt{25a^2b^2}$, $\sqrt[3]{8a^3b^3c^3}$.
8. $\sqrt{18}$, $\sqrt{28}$, $\sqrt[3]{27}$.
9. $\sqrt{\frac{8}{3}}$, $\sqrt{\frac{27}{8}}$, $\sqrt{4\frac{4}{15}}$, $\sqrt{5\frac{5}{24}}$.
10. $3\frac{1}{2}\sqrt{24} + 5\frac{3}{4}\sqrt{54} + 13\frac{1}{3}\sqrt{99}$.
11. $\sqrt[3]{24a^3}$, $\sqrt[3]{81a^3}$, $\sqrt[3]{192a^3}$.
12. $2\sqrt[3]{40}$, $3\sqrt[3]{135}$, $\sqrt[3]{5000}$.

13. $\sqrt[3]{32\,a^4b^4c}$, $\sqrt[3]{108\,ab^4c^4}$, $\sqrt[3]{500\,a^4bc^4}$.

14. $\sqrt[5]{a^6b^7c^3}$, $\sqrt[5]{ab^7c^8}$, $\sqrt[5]{a^6b^2c^8}$.

15. $\sqrt[n]{a^{2n}b^n}$, $\sqrt[n]{a^{4n}b^{2n}}$, $\sqrt[n]{a^{3n}b^{5n}}$.

16. $\sqrt[n+3]{a^{n+3}b^{2(n+3)}}$, $\sqrt[2n+6]{a^{2n+6}b^{4n+12}}$.

17. $\sqrt[x]{m^{2x+2}}$, $\sqrt[2x]{m^{4x+4}}$, $\sqrt[3x]{m^{6x+6}}$.

18. $\sqrt[m]{a^mb^n}$, $\sqrt[m]{a^{2m}b^{mn}}$, $\sqrt[m]{a^{3m}b^{mn+2}c^m}$.

CHAPTER XX.

SUBTRACTION OF RADICALS.

226. It is evident from the laws of subtraction that $a^{\frac{1}{2}}$ minus $a^{\frac{1}{3}}$ can only be indicated as the terms $a^{\frac{1}{2}}$ and $a^{\frac{1}{3}}$ are dissimilar, but $a^{\frac{1}{2}}$ minus $a^{\frac{1}{2}}$, being similar terms, can be subtracted directly by subtracting the coefficient of the subtrahend from that of the minuend.

Thus, $$a^{\frac{1}{2}} - a^{\frac{1}{2}} = 0.$$

227. *When radicals are similar, one may be subtracted from another by subtracting the coefficient of the subtrahend from that of the minuend and prefixing the result to the common surd factor.*

228. *When the radicals are dissimilar, subtraction must be indicated by the sign minus.*

Examples:

(1) From $\sqrt[3]{81}$ subtract $\sqrt[3]{24}$.

Now, $$\sqrt[3]{81} = 3\sqrt[3]{3},$$
$$\sqrt[3]{24} = 2\sqrt[3]{3}.$$
$$\therefore\ 3\sqrt[3]{3} - 2\sqrt[3]{3} = (3-2)\sqrt[3]{3} = \sqrt[3]{3}.$$

(2) From $\sqrt{48} + \sqrt{1728}$ subtract $\sqrt{108}$.

Now, $$\sqrt{48} = 2\sqrt{12},$$
$$\sqrt{1728} = 12\sqrt{12},$$
$$\sqrt{108} = 3\sqrt{12}.$$
$$\therefore\ (2+12)\sqrt{12} - 3\sqrt{12} = (14-3)\sqrt{12} = 11\sqrt{12}.$$

Perform the operation indicated:

(3) $\sqrt[n]{a^{n+2}b^{n+3}} - \sqrt[n]{a^{n+3}b^{n+2}}$.

Now,
$$\sqrt[n]{a^{n+2}b^{n+3}} = ab\sqrt[n]{a^2b^3},$$
$$\sqrt[n]{a^{n+3}b^{n+2}} = ab\sqrt[n]{a^3b^2}.$$
$$\therefore \sqrt[n]{a^{n+2}b^{n+3}} - \sqrt[n]{a^{n+3}b^{n+2}} = ab\left(\sqrt[n]{a^2b^3} - \sqrt[n]{a^3b^2}\right).$$

(4) From $\sqrt{\frac{1}{2}} + \sqrt{\frac{2}{9}}$ subtract $\sqrt{\frac{1}{50}}$.

Now,
$$\sqrt{\tfrac{1}{2}} = \sqrt{\tfrac{2}{4}} = \tfrac{1}{2}\sqrt{2},$$
$$\sqrt{\tfrac{2}{9}} = \sqrt{\tfrac{2}{9}} = \tfrac{1}{3}\sqrt{2},$$
$$\sqrt{\tfrac{1}{50}} = \sqrt{\tfrac{2}{100}} = \tfrac{1}{10}\sqrt{2}.$$
$$\begin{aligned}\therefore \left(\sqrt{\tfrac{1}{2}} + \sqrt{\tfrac{1}{9}}\right) - \sqrt{\tfrac{1}{50}} &= \left(\tfrac{1}{2} + \tfrac{1}{3}\right)\sqrt{2} - \tfrac{1}{10}\sqrt{2} \\ &= \tfrac{5}{6}\sqrt{2} - \tfrac{1}{10}\sqrt{2} \\ &= \tfrac{25}{30}\sqrt{2} - \tfrac{3}{30}\sqrt{2} \\ &= \tfrac{22}{30}\sqrt{2} = \tfrac{11}{15}\sqrt{2}.\end{aligned}$$

EXERCISE 81.

Perform the operations indicated:

1. $\sqrt{18} - \sqrt{8}$.
2. $\sqrt[3]{108} - \sqrt[3]{24}$.
3. $\sqrt[5]{64} - \frac{1}{2}\sqrt[5]{729}$.
4. $\sqrt[4]{243} - 2\sqrt[4]{3}$.
5. $\sqrt{20} - \frac{1}{3}\sqrt{45}$.
6. $\sqrt{200} - \sqrt[4]{64}$.
7. $\sqrt{18} + \sqrt{28} - \sqrt{75}$.
8. $\sqrt{72} + \sqrt{288} - \sqrt[3]{3456}$.
9. $\sqrt{a^3} - \sqrt{a}$.
10. $\sqrt{\dfrac{1}{a}} - \sqrt{\dfrac{1}{a^3}}$.
11. $\sqrt[3]{16a^5b^3} - \sqrt[3]{54a^3b^3}$.
12. $\sqrt{4x^3} - \sqrt{9x^5}$.
13. $\sqrt[n]{4^na^{2n}b^n} - \sqrt[n]{2^na^{2n}b^n}$.
14. $\sqrt[3]{81a^4} - \sqrt[3]{24a^4}$.
15. $\sqrt[2n]{x^{2mn}a^{3n}} - \sqrt[2n]{x^{4mn}a^{5n}}$.
16. $\sqrt{a^2 + 2ab + b^2} - \sqrt{a^2 - 2ab + b^2}$.

CHAPTER XXI.

MULTIPLICATION OF RADICALS.

229. Since by (214) $a\sqrt[n]{a} \times \sqrt[n]{b} = a\sqrt[n]{ab}$, it follows *that radicals of a common index may be multiplied together as integers, and the result placed under a radical with the common index, the product of the coefficient being prefixed.*

EXAMPLES:

(1) Multiply $\sqrt{2}$ by $\frac{1}{2}\sqrt{3}$.

$$\begin{aligned}\sqrt{2} \times \tfrac{1}{2}\sqrt{3} &= 1 \times \tfrac{1}{2} \times \sqrt{2} \times \sqrt{3}\\ &= \tfrac{1}{2} \times \sqrt{6}\\ &= \tfrac{1}{2}\sqrt{6}.\end{aligned}$$

(2) Multiply $3\sqrt{8}$ by $5\sqrt{18}$.

$$\begin{aligned}3\sqrt{8} \times 5\sqrt{18} &= 6\sqrt{2} \times 15\sqrt{2}\\ &= 6 \times 15 \times \sqrt{2} \times \sqrt{2}\\ &= 90\sqrt{4}\\ &= 180.\end{aligned}$$

(3) Multiply $\frac{1}{2}\sqrt{\frac{1}{2}}$ by $\frac{1}{3}\sqrt{\frac{1}{3}}$ by $\frac{1}{4}\sqrt{\frac{1}{4}}$.

$$\begin{aligned}\tfrac{1}{2}\sqrt{\tfrac{1}{2}} \times \tfrac{1}{3}\sqrt{\tfrac{1}{3}} \times \tfrac{1}{4}\sqrt{\tfrac{1}{4}} &= \tfrac{1}{2}\sqrt{\tfrac{1}{2}} \times \tfrac{1}{3}\sqrt{\tfrac{1}{3}} \times \tfrac{1}{8}\\ &= \tfrac{1}{2} \times \tfrac{1}{3} \times \tfrac{1}{8} \times \sqrt{\tfrac{1}{2}} \times \sqrt{\tfrac{1}{3}}\\ &= \tfrac{1}{48} \times \sqrt{\tfrac{1}{6}}\\ &= \tfrac{1}{48} \times \tfrac{1}{6}\sqrt{6}\\ &= \tfrac{1}{288}\sqrt{6}.\end{aligned}$$

(4) Multiply together

$$\sqrt{\frac{1}{a+b}},\ \sqrt{\frac{1}{a-b}},\ \text{and}\ (a^2-b^2)\sqrt{a^2-b^2}.$$

Now,

$$\sqrt{\frac{1}{a+b}}=\frac{1}{a+b}\sqrt{a+b},$$

$$\sqrt{\frac{1}{a-b}}=\frac{1}{a-b}\sqrt{a-b},$$

$$(a^2-b^2)\sqrt{a^2-b^2}=(a^2-b^2)\sqrt{a^2-b^2},$$

and we have:

$$\begin{aligned}\frac{1}{a+b}\times\frac{1}{a-b}\times(a^2-b^2)\times\sqrt{a+b}\times\sqrt{a-b}\times\sqrt{a^2-b^2},\\ =\frac{a^2-b^2}{a^2-b^2}\sqrt{(a^2-b^2)^2}\\ =a^2-b^2.\end{aligned}$$

(5) Multiply:

$$\begin{array}{r}\sqrt{a}\ -\sqrt{b}\quad\quad\\ \underline{\sqrt{a}\ -\sqrt{b}\quad\quad}\\ \sqrt{a^2}-\sqrt{ab}\quad\quad\\ \underline{-\sqrt{ab}+\sqrt{b^2}}\\ a-2\sqrt{ab}+b.\end{array}$$

(6) Multiply:

$$\begin{array}{r}a\sqrt{a}-\sqrt{b}\qquad\qquad\qquad\\ \underline{a^2\sqrt{b}-b^2\sqrt{a}}\qquad\qquad\quad\\ a^3\sqrt{ab}-a^2\sqrt{b^2}\qquad\qquad\qquad\\ \underline{-ab^2\sqrt{a^2}+b^2\sqrt{ab}}\\ a^3\sqrt{ab}-a^2\sqrt{b^2}-ab^2\sqrt{a^2}+b^2\sqrt{ab}\\ =(a^3+b^2)\sqrt{ab}-a^2b-a^2b^2.\end{array}$$

EXERCISE 82.

Perform the operations indicated:

1. $\sqrt{2} \times 2\sqrt{3}$.

2. $3\sqrt{5} \times 3\sqrt{20}$.

3. $4\sqrt{5} \times 5\sqrt{6} \times 6\sqrt{30}$.

4. $a\sqrt{b} \times b\sqrt{c} \times c\sqrt{a}$.

5. $a\sqrt{bc} \times b\sqrt{ac} \times c\sqrt{ab}$.

6. $\frac{1}{2}\sqrt{3} \times \frac{1}{3}\sqrt{4} \times \frac{1}{4}\sqrt{5}$.

7. $\frac{1}{2}\sqrt[3]{\frac{1}{32}x^4} \times \frac{1}{3}\sqrt[3]{\frac{1}{108}y^4} \times \frac{1}{3}\sqrt{81z^4}$.

8. $\frac{1}{a}\sqrt{\frac{1}{x^3}} \times \frac{1}{b}\sqrt[3]{\frac{1}{y^4}} \times \frac{1}{c}\sqrt[6]{\frac{1}{z^7}}$.

9. $(\sqrt{2} + \sqrt{3}) \times \sqrt{4}$.

10. $(\frac{1}{2}\sqrt{2} - \frac{1}{3}\sqrt{3}) \times \frac{1}{4}\sqrt{4}$.

11. $(\sqrt[3]{3} + \sqrt[4]{4}) \times \sqrt[5]{2}$.

12. $\left(\frac{a}{b}\sqrt[3]{c} + \frac{c}{b}\sqrt[4]{a}\right) \times \frac{a^2}{c^2}\sqrt{ac}$.

13. $(\sqrt{2} + 1) \times (\sqrt{2} - 1)$.

14. $(a\sqrt{a} - 1) \times (a\sqrt{a} + 1)$.

15. $(x^2\sqrt{y} + y^2\sqrt{x}) \times (x^2\sqrt{y} - y^2\sqrt{x})$.

16. $\left(\frac{1}{a}\sqrt{\frac{1}{b}} + \sqrt{\frac{1}{c}}\right) \times \left(\frac{1}{a}\sqrt{\frac{1}{b}} + \sqrt{\frac{1}{c}}\right)$.

17. $(x + \sqrt{xy} + y) \times (x - \sqrt{xy} + y)$.

18. $(\sqrt[3]{a} - \sqrt[3]{b}) \times (\sqrt[3]{a^2} + \sqrt[3]{ab} + \sqrt[3]{b^2})$.

19. $(\sqrt[3]{a} + \sqrt[3]{b}) \times (\sqrt[3]{a^2} - \sqrt[3]{ab} + \sqrt[3]{b^2})$.

CHAPTER XXII.

DIVISION OF RADICALS.

Since $$a\sqrt[n]{ab} = a\sqrt[n]{a} \times \sqrt[n]{b},$$
we have, by (214), $$a\sqrt[n]{ab} \div a\sqrt[n]{a} = \sqrt[n]{b}.$$

230. Hence, it follows *that one radical may be divided by another, if both are of common index, by dividing the coefficient of the divisor into the coefficient of the dividend for the coefficient of the quotient, and the surd factor of the divisor into the surd factor of the dividend for the surd factor of the quotient.*

EXAMPLES:

(1) Divide $4\sqrt{8}$ by $6\sqrt{32}$.

Now, $$4\sqrt{8} = 8\sqrt{2};$$
$$3\sqrt{32} = 12\sqrt{2}.$$
$$\therefore\ 8\sqrt{2} \div 12\sqrt{2} = \tfrac{8}{12} = \tfrac{2}{3}.$$

(2) Divide $a^2\sqrt{(a+b)^3}$ by $b^2\sqrt{(a-b)^3}$.

Now, $$a^2\sqrt{(a+b)^3} = a^2(a+b)\sqrt{a+b},$$
and $$b^2\sqrt{(a-b)^3} = b^2(a-b)\sqrt{a-b}.$$
Hence, $$\frac{a^2(a+b)}{b^2(a-b)}\sqrt{\frac{a+b}{a-b}};$$
but $$\sqrt{\frac{a+b}{a-b}} = \frac{1}{a-b}\sqrt{a^2-b^2}.$$
$$\therefore\ \frac{a^2(a+b)}{b^2(a-b)}\sqrt{\frac{a+b}{a-b}} = \frac{a^2(a+b)}{b^2(a-b)^2}\sqrt{a^2-b^2}.$$

(3) Divide $a\sqrt{b} - b\sqrt{a}$ by $\sqrt{ab}$.

Dividing each term of the dividend separately, we have,

$$\frac{a\sqrt{b}}{\sqrt{ab}} - \frac{b\sqrt{a}}{\sqrt{ab}} = a\sqrt{\frac{b}{ab}} - b\sqrt{\frac{a}{ab}}$$

$$= a\sqrt{\frac{1}{a}} - b\sqrt{\frac{1}{b}}$$

$$= \sqrt{a} - \sqrt{b}.$$

TO RATIONALIZE THE DIVISOR.

231. This method is employed when the divisor is a compound expression. Also when it is desired to find the approximate numerical value of one surd divided by another.

EXAMPLES:

(1) Find the approximate value of $\frac{\sqrt{3}}{\sqrt{5}}$.

Multiplying both terms by $\sqrt{5}$, we rationalize the divisor, and have,

$$\frac{\sqrt{15}}{5} = \tfrac{1}{5}\sqrt{15} = \frac{3.872}{5} = .774+.$$

(2) Divide $2 + 3\sqrt{2}$ by $2 - 3\sqrt{2}$.

Multiplying both dividend and divisor by $2+3\sqrt{2}$, we have,

$$\frac{2 + 3\sqrt{2}}{2 - 3\sqrt{2}} = \frac{(2 + 3\sqrt{2})^2}{4 - 18}$$

$$= \frac{4 + 12\sqrt{2} + 18}{4 - 18}$$

$$= \frac{22 + 12\sqrt{2}}{-14}$$

$$= \frac{11 + 6\sqrt{2}}{-7}$$

$$= -\frac{11 + 6\sqrt{2}}{7}$$

EXERCISE 83.

Divide:

1. $\sqrt{360}$ by $\sqrt{10}$.
2. $\sqrt{8}$ by $\sqrt{64}$.
3. $\sqrt[3]{256}$ by $\sqrt[3]{108}$.
4. $\sqrt[3]{a^2}$ by $\sqrt[3]{a}$.
5. $3\sqrt{\frac{1}{2}}$ by $2\sqrt{\frac{1}{3}}$.
6. $a^2\sqrt{b}$ by $b^2\sqrt{a}$.
7. $x^5\sqrt[3]{y^4}$ by $x^3\sqrt[3]{y^2}$.
8. $3\sqrt{a}$ by $\sqrt{a}$.
9. $9a\sqrt{b}$ by $3a\sqrt{b}$.
10. $\sqrt[3]{\frac{5}{16}}$ by $\sqrt{\frac{5}{16}}$.
11. $\sqrt{\dfrac{a}{x}}$ by $\sqrt[3]{\dfrac{x}{a}}$.
12. $\sqrt[n]{a^2}$ by $\sqrt[n]{a}$.
13. $\sqrt[n+1]{a^2b^{2(n+1)}}$ by $\sqrt[n+1]{a^3c^{3(n+1)}}$.
14. $\sqrt{a^{12}}+\sqrt{a^{15}}+\sqrt{a^{21}}$ by $\sqrt{a}$.
15. $a\sqrt{x^3}-b\sqrt{x^5y^3}+c\sqrt{x^7y^5}$ by $abc\sqrt{x^3y}$.
16. $21\,ax\sqrt{yz}-7\,axy\sqrt{xz}+14\,a\sqrt{z^2}$ by $7\,a\sqrt{z}$.

EXERCISE 84.

Find the approximate value of:

1. $\dfrac{1}{\sqrt{2}}$.
2. $\dfrac{1}{\sqrt{3}}$.
3. $\dfrac{2}{\sqrt{5}}$.
4. $\dfrac{3}{\sqrt[3]{7}}$.
5. $\sqrt{\frac{2}{3}}$.
6. $\sqrt{\frac{3}{5}}$.
7. $\sqrt{\frac{5}{12}}$.
8. $\sqrt{\frac{9}{13}}$.
9. $\frac{3}{5}\sqrt{\frac{2}{3}}$.
10. $\dfrac{1}{\frac{3}{2}\sqrt{2}}$.
11. $\dfrac{2}{\sqrt[3]{5}}$.
12. $\sqrt[3]{\frac{5}{7}}$.
13. $\sqrt[3]{\frac{7}{12}}$.
14. $\sqrt[3]{\frac{7}{16}}$.
15. $\frac{1}{2}\sqrt[3]{\frac{3}{7}}$.
16. $\frac{5}{17}\sqrt{\frac{3}{5}}$.
17. $\dfrac{3\frac{1}{2}}{\sqrt[3]{\frac{2}{3}}}$.

EXERCISE 85.

Rationalize the denominators in the following:

1. $\frac{1}{\sqrt{a+b}}$.

2. $\frac{a-b}{\sqrt{a+b}}$.

3. $\frac{3}{\sqrt{b-c}}$.

4. $\frac{4}{\sqrt{a+b+c}}$.

5. $\frac{5}{\sqrt{3-2}}$.

6. $\frac{3+2\sqrt{2}}{3-2\sqrt{2}}$.

7. $\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$.

8. $\frac{a+a\sqrt{b}}{x-y\sqrt{z}}$.

9. $\frac{1}{a-\sqrt{a-b}}$.

10. $\frac{3}{x+\sqrt{x^2+1}}$.

11. $\frac{5}{x-\sqrt{x^2+x+1}}$.

12. $\frac{7}{\sqrt{5}+\sqrt{6}-\sqrt{7}}$.

13. $\frac{3}{3\sqrt{2}-2\sqrt{3}+6\sqrt{5}}$.

14. $\frac{7}{5\sqrt{7}+3\sqrt{5}-7\sqrt{3}}$.

15. $\frac{\sqrt{a+b}-\sqrt{c}}{\sqrt{a+b}+\sqrt{c}}$.

16. $\frac{\sqrt{x+y}+\sqrt{x-y}}{\sqrt{x+y}-\sqrt{x-y}}$.

CHAPTER XXIII.

INVOLUTION AND EVOLUTION OF RADICALS.

232. The power or the root of a radical may be found by using fractional exponents.

Since $$a^{\frac{1}{2}} = \sqrt{a},$$

conversely $$\sqrt{a} = a^{\frac{1}{2}},$$

and $$\sqrt{\sqrt{a}} = (\sqrt{a})^{\frac{1}{2}},$$

and $$\sqrt{\sqrt{\sqrt{a}}} = \left(\sqrt{\sqrt{a}}\right)^{\frac{1}{2}}.$$

Now,
$$\begin{aligned}(\sqrt{a})^{\frac{1}{2}} &= \sqrt{a^{\frac{1}{2}}}\\ &= \left((a)^{\frac{1}{2}}\right)^{\frac{1}{2}}\\ &= (a)^{\frac{1}{4}}.\end{aligned}$$

$$\therefore\ \sqrt{\sqrt{a}} = a^{\frac{1}{4}}.$$

In like manner, $$\sqrt{\sqrt{\sqrt{a}}} = a^{\frac{1}{8}}.$$

Again,
$$\begin{aligned}(\sqrt{a})^2 &= (a^{\frac{1}{2}})^2\\ &= a^{\frac{2}{2}}\\ &= a.\end{aligned}$$

And
$$\begin{aligned}(\sqrt[3]{a^2})^3 &= (a^{\frac{2}{3}})^3\\ &= a^{\frac{6}{3}}\\ &= a^2.\end{aligned}$$

In like manner, $(4a\sqrt{b})^2 = 4^2a^2(b^{\frac{1}{2}})^2$

$$= 16\,a^2b.$$

And $\sqrt{4a^2\sqrt{b}} = 4^{\frac{1}{2}}(a^2)^{\frac{1}{2}}(b^{\frac{1}{2}})^{\frac{1}{2}}$

$$= 2\,ab^{\frac{1}{4}}$$

$$= 2\,a\sqrt[4]{b}.$$

EXERCISE 86.

Perform the operations indicated:

1. $\sqrt[3]{8^2}$.
2. $\sqrt[3]{(\frac{1}{8})^2}$.
3. $\sqrt[9]{27^3}$.
4. $\sqrt{\sqrt{81}}$.
5. $\sqrt{\sqrt{64}}$.
6. $\sqrt{\sqrt{\sqrt{256}}}$.
7. $(\frac{1}{2}\sqrt{(\frac{1}{4})^3})^2$.
8. $(\frac{1}{8}\sqrt{272})^4$.
9. $(9\,a^2b^2\sqrt{x^{12}})^{\frac{1}{2}}$.
10. $(3\,x^2y\sqrt[3]{\sqrt{64\,a^{12}}})^2$.
11. $(64\,x^4y^8\,\sqrt[3]{a^9})^{\frac{1}{3}}$.
12. $(4\sqrt[3]{2\sqrt{16\,a^6}})^{\frac{1}{2}}$.
13. $\sqrt{\sqrt{(a^4-1)^4}}$.
14. $\{\frac{1}{4}(a^2+2\,ab+b^2)^{\frac{1}{2}}\}^2$.
15. $\sqrt[4]{(a^2-2\,ab+b^2)^2}$.
16. $\sqrt{\sqrt{\left(\frac{x-y}{x+y}\right)^8\left(\frac{x+y}{x-y}\right)^4}}$.

QUADRATIC SURDS.

233. *The product or quotient of two quadratic surds which have not the same irrational part will be irrational.*

For $\sqrt{xy} \times \sqrt{xz} = \sqrt{x^2yz} = x\sqrt{yz}$,

and $x\sqrt{yz} \div \sqrt{xz} = \sqrt{xy}$.

234. *One quadratic surd cannot be the sum or the difference of two others which have not the same irrational part.*

For, suppose that $\sqrt{x} = \sqrt{a} + \sqrt{b}$.

Squaring, $x = a + 2\sqrt{ab} + b$,

and $\frac{1}{2}(x - a - b) = \sqrt{ab}$.

That is, a surd equals a rational number, which is impossible.

In the equation $a \pm \sqrt{b} = x \pm \sqrt{y}$, $a = x$ *and* $\sqrt{b} = \sqrt{y}$.

If a does not equal x, suppose it equals $x \pm m$.

Then $$x \pm m \pm \sqrt{b} = x \pm \sqrt{y},$$

and $$\pm m \pm \sqrt{b} = \pm \sqrt{y},$$

which is impossible.

235. *The square root of a binomial, one of whose terms is a quadratic surd and the other rational, may sometimes be expressed by a binomial, one or both terms of which may be a quadratic surd.*

Let $a + \sqrt{b}$ be the given binomial, and let its square root be expressed by $\sqrt{x} + \sqrt{y}$.

Then, $$\sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y}. \qquad (1)$$

Squaring, $$a + \sqrt{b} = x + 2\sqrt{xy} + y. \qquad (2)$$

Hence, $$a = x + y, \qquad (3)$$

and $$\sqrt{b} = 2\sqrt{xy}. \qquad (4)$$

Subtracting (4) from (3),

$$a - \sqrt{b} = x - 2\sqrt{xy} + y. \qquad (5)$$

Extracting the square root of (5),

$$\sqrt{a - \sqrt{b}} = \sqrt{x} - \sqrt{y}. \qquad (6)$$

Multiplying (1) by (6),

$$\sqrt{a^2 - b} = x - y. \qquad (7)$$

Adding (3) and (7),

$$a + \sqrt{a^2 - b} = 2x. \qquad (8)$$

Hence, $$\sqrt{\frac{a + \sqrt{a^2 - b}}{2}} = \sqrt{x}. \qquad (9)$$

Subtracting (7) from (3),

$$a - \sqrt{a^2 - b} = 2y. \qquad (10)$$

Hence, $$\sqrt{\frac{a-\sqrt{a^2-b}}{2}}=\sqrt{y}. \tag{11}$$

$$\therefore\ \sqrt{x}\pm\sqrt{y}=\sqrt{\frac{a+\sqrt{a^2-b}}{2}}\pm\sqrt{\frac{a-\sqrt{a^2-b}}{2}};$$

$$\therefore\ \sqrt{a\pm\sqrt{b}}=\sqrt{\frac{a+\sqrt{a^2-b}}{2}}\pm\sqrt{\frac{a^2-\sqrt{a^2-b}}{2}}.$$

EXAMPLES:

(1) Extract the square root of $14+6\sqrt{5}$.

Now, $$\sqrt{x}+\sqrt{y}=\sqrt{14+6\sqrt{5}}, \tag{1}$$

and $$\sqrt{x}-\sqrt{y}=\sqrt{14-6\sqrt{5}}. \tag{2}$$

Hence, $$x-y=\sqrt{196-180}. \tag{3}$$

$$\therefore\ x-y=4. \tag{4}$$

Squaring (1),

$$x+2\sqrt{xy}+y=14+6\sqrt{5}. \tag{5}$$

Hence, $$x+y=14. \tag{6}$$

Adding (4) and (6), $$2x=18.$$

$$\therefore\ x=\ 9.$$

Subtracting (4) from (6), $$2y=10.$$

$$\therefore\ y=\ 5.$$

$$\therefore\ \sqrt{x}+\sqrt{y}=\sqrt{9}+\sqrt{5}$$
$$=3+\sqrt{5}.$$

Substituting the values of a and $\sqrt{b}$ of the last example in the given formula, we have,

$$\sqrt{a+\sqrt{b}}\quad=\sqrt{\frac{a+\sqrt{a^2-b}}{2}}+\sqrt{\frac{a-\sqrt{a^2-b}}{2}},$$

$$\sqrt{14+6\sqrt{5}}=\sqrt{\frac{14+\sqrt{196-180}}{2}}+\sqrt{\frac{14-\sqrt{196-180}}{2}},$$
$$=\sqrt{\frac{14+4}{2}}+\sqrt{\frac{14-4}{2}},$$
$$=3+\sqrt{5}.$$

236. *If the given binomial is expressed in the form* $a+2\sqrt{b}$, *the answer may be found by inspection.*

EXAMPLE:

(1) Find the square root of $24-2\sqrt{143}$.

It remains to find two surds whose product gives $\sqrt{143}$, and the sum of whose squares gives 24. These are $\sqrt{11}$ and $\sqrt{13}$.

Hence the square root of $24-2\sqrt{143}=\sqrt{11}-\sqrt{13}$.

Again:

(2) Find the square root of $11-6\sqrt{2}$.

Reducing to the form $a+2\sqrt{b}$, we have, $11-2\sqrt{18}$.

It remains to find, therefore, two surds whose product gives $\sqrt{18}$, and the sum of whose squares gives 11.

They are $\sqrt{9}$ and $\sqrt{2}$.

Hence the square root of

$$11-6\sqrt{2}=\sqrt{9}-\sqrt{2},$$
$$=3-\sqrt{2}.$$

EXERCISE 87.

Extract the square root of the following:

1 $8+2\sqrt{15}$.

2. $21-14\sqrt{2}$.

3. $16+2\sqrt{15}$.

4. $30+2\sqrt{216}$.

5. $30-\sqrt{116}$.

6. $117+8\sqrt{101}$.

7. $57-12\sqrt{15}$.

8. $4+\sqrt{15}$.

9. $ab(a+b)+2\sqrt{a^3b^3}$.

10. $4(a^2x+b^2y)-8\sqrt{a^2b^2xy}$.

11. $9\{m^2(a+b)+n^2(a-b)\}-18mn\sqrt{a^2-b^2}$.

ADDITIONAL EXAMPLES.

EXERCISE 88.

Simplify:

1. $\sqrt[4]{256}$.
2. $\sqrt[4]{(a^2+2ab+b^2)^2}$.
3. $\sqrt[6]{(1728x^6y^{12})^3}$.
4. $\sqrt[12]{(81x^2y)^4}$.
5. $\sqrt{125}$.
6. $\sqrt[3]{375}$.
7. $\sqrt[3]{216m^4}$.
8. $\sqrt[3]{-1728}$.
9. $\sqrt[3]{\frac{125}{256}}$.
10. $\sqrt{\frac{75}{16}x^2y^2}$.
11. $\sqrt[4]{1875}$.
12. $6\sqrt[3]{81}$.
13. $7\sqrt[4]{(125)^2}$.
14. $\sqrt{\frac{1}{5}}$.
15. $\sqrt{\frac{2}{3}}$.
16. $\sqrt[4]{\frac{2}{32}}$.
17. $\sqrt[5]{\frac{1}{64}}$.
18. $\sqrt[6]{\frac{1}{256}}$.
19. $\sqrt[3]{\frac{25}{36}}$.
20. $\sqrt[4]{\frac{144}{1728}}$.
21. $\sqrt[3]{\frac{a^2b^2}{4}}$.
22. $\sqrt[4]{\frac{a^3b^3}{8}}$.
23. $\sqrt[3]{\frac{9a^2}{125x}}$.
24. $\sqrt[3]{\frac{a+b}{(a-b)^2}}$.
25. $\sqrt[4]{\frac{(a+b)^2}{(a+c)^3}}$.

Express as entire surds:

26. $3\sqrt{3}$.
27. $2\sqrt{21}$.
28. $9\sqrt{a}$.
29. $3\sqrt[3]{3}$.
30. $4\sqrt[4]{4}$.
31. $-5\sqrt[3]{2}$.
32. $-a\sqrt[5]{m}$.
33. $-bc\sqrt[3]{2}$.
34. $-\frac{1}{3}\sqrt[3]{3}$.
35. $-2\sqrt[7]{x^2}$.

Reduce to surds of same order:

36. $\sqrt{2}, \sqrt[3]{3}$.

37. $\sqrt[3]{7}, \sqrt[4]{2}$.

38. $\sqrt{5}, \sqrt[3]{a}$.

39. $-2\sqrt[3]{m}, \ -2\sqrt[4]{pq}$.

40. $m\sqrt[5]{2}, \ x\sqrt[3]{a}$.

41. $a^{\frac{1}{2}}, b^{\frac{1}{2}}, c^{\frac{1}{3}}, d^{\frac{1}{4}}$.

42. $2^{\frac{1}{a}}, 3^{\frac{1}{b}}, 4^{\frac{1}{c}}$.

43. $\sqrt[n]{2}, \sqrt[m]{3}$.

44. $\sqrt[x]{m}, \sqrt[y]{n}$.

45. $3\,a\sqrt[p]{b}, \ 3\,c\sqrt[q]{a}$.

46. Add $\sqrt{3}, \ \sqrt[3]{4}, \ \sqrt{5}$.

47. Add $-\sqrt{2}, \ \sqrt[3]{a}, \ \sqrt[6]{4\,a^2}$.

48. Add $2\sqrt{a} - 5\sqrt[4]{a^2} + 10\sqrt[8]{a^4}$.

49. Subtract $\sqrt{75} - \sqrt{27}$.

50. Subtract $\sqrt[3]{(a-b)^4} - \sqrt[3]{(a^2-b^2)(a+b)^4}$.

51. Multiply $\sqrt{x} - \sqrt{y}$ and $\sqrt{2\,x} + \sqrt{3\,y}$.

52. Multiply $\sqrt{a^2} - \sqrt{b^2}$ and $\sqrt{a} - \sqrt{b}$.

53. Multiply $(\sqrt{2} - \sqrt{a} - \sqrt{b})$ by $(\sqrt{2} + \sqrt{a} - \sqrt{b})$.

54. Expand $(2\sqrt{x} - 3\sqrt{y} + 4\sqrt{z})^2$.

55. Expand $(\sqrt{a-b} - \sqrt{a+b})^3$.

56. Divide $\sqrt{a} + \sqrt{b}$ by $\sqrt{a} - \sqrt{b}$.

57. Divide $\sqrt{a}$ by $\sqrt{m} + \sqrt{n} + \sqrt{p}$.

58. Divide $\sqrt{2} + \sqrt{3}$ by $\sqrt{3} + \sqrt{4} - \sqrt{5}$.

CHAPTER XXIV.

IMAGINARY EXPRESSIONS.

237. The indicated even root of a negative quantity is called an **Imaginary Expression.**

238. Although the symbol for the even root of a negative quantity represents an impossibility, imaginary expressions are frequently useful in mathematical investigations, as we shall hereafter show.

Let m be any real quantity; then the square roots of $-m^2$ will be $\pm\sqrt{-m^2}$.

If we regard $-m^2$ as the product of m^2 and -1, we may further simplify the expression $\pm\sqrt{-m^2}$ into $\pm m\sqrt{-1}$.

239. Every imaginary expression may be made to assume the form $m+n\sqrt{-1}$. If in the expression $m+n\sqrt{-1}$, m and n are real quantities, the expression is still regarded as an imaginary. If in this expression $m+n\sqrt{-1}$, $n=0$, the real part only remains; if $m=0$, the imaginary part only remains.

240. $\boldsymbol{m+n\sqrt{-1}}$ is regarded as the *typical form* for an imaginary expression.

$\boldsymbol{n\sqrt{-1}}$ is called a **Pure Imaginary.**

$\boldsymbol{m+n\sqrt{-1}}$ and $\boldsymbol{m-n\sqrt{-1}}$ are called **Conjugate Imaginaries.**

241. As the symbol $\sqrt{-1}$, the *imaginary unit*, is the square root of -1, we have,

$$\sqrt{-1}\times\sqrt{-1}=-1,$$

and

$$(\sqrt{-1})^2\times\sqrt{-1}=-\sqrt{-1},$$

and

$$(\sqrt{-1})^2\times(\sqrt{-1})^2=+1.$$

242. Hence,

$$(\sqrt{-1})^1=\sqrt{-1},$$
$$(\sqrt{-1})^2=-1,$$
$$(\sqrt{-1})^3=-\sqrt{-1},$$
$$(\sqrt{-1})^4=+1.$$

Following this multiplication to higher powers, we shall find the fifth power the same as the first, the sixth the same as the second, and so on.

243. And in general, if n is a positive integer, we have,

$$(\sqrt{-1})^{4n} \quad =1,$$
$$(\sqrt{-1})^{4n+1}=\sqrt{-1},$$
$$(\sqrt{-1})^{4n+2}=-1,$$
$$(\sqrt{-1})^{4n+3}=-\sqrt{-1}.$$

If in the equation $m+n\sqrt{-1}=a+b\sqrt{-1}$, m, n, a, and b are real numbers, $m=a$, and $n=b$.

For, transposing, we have,

$$m-a=(b-n)\sqrt{-1}.$$

That is, a real number equals a pure imaginary, which is impossible unless $m-a=0$; and $b-n=0$.

Hence, $m=a$, and $n=b$.

EXAMPLES:

(1) Add together $x+y\sqrt{-1}$ and $x-y\sqrt{-1}$.

$$\begin{array}{r} x+y\sqrt{-1} \\ x-y\sqrt{-1} \\ \hline 2x \qquad\quad \end{array}$$

Hence the sum of conjugate imaginaries is real.

(2) Multiply $x + y\sqrt{-1}$ by $x - y\sqrt{-1}$.

$$\begin{array}{l} x + y\sqrt{-1} \\ \underline{x - y\sqrt{-1}} \\ x^2 + y^2 \end{array}$$

Hence the product of conjugate imaginaries is real.

(3) Add together $18 + 3\sqrt{-1}$ and $17 - 2\sqrt{-1}$.

$$\begin{array}{l} 18 + 3\sqrt{-1} \\ \underline{17 - 2\sqrt{-1}} \\ 35 + \ \ \sqrt{-1} \end{array}$$

(4) Square $6 - 5\sqrt{-1}$.

By the formula, we have,

$$\begin{aligned} (6 - 5\sqrt{-1})^2 &= 36 - 60\sqrt{-1} + 25(\sqrt{-1})^2 \\ &= 36 - 60\sqrt{-1} - 25 \\ &= 11 - 60\sqrt{-1}. \end{aligned}$$

(5) Extract the square root of $-10 + 2\sqrt{24}$.

$$\begin{aligned} \sqrt{-10 + 2\sqrt{24}} &= \sqrt{-1}\sqrt{10 - 2\sqrt{24}} \\ &= \sqrt{-1}\sqrt{6 - 2\sqrt{24} + 4} \\ &= \sqrt{-1}(\sqrt{6} - \sqrt{4}) \\ &= \sqrt{-1}(\sqrt{6} - 2) \\ &= \sqrt{-6} - 2\sqrt{-1}. \end{aligned}$$

(6) Divide $10 - 3\sqrt{-1}$ by $9 - 4\sqrt{-1}$.

$$\begin{aligned} \frac{10 - 3\sqrt{-1}}{9 - 4\sqrt{-1}} &= \frac{(10 - 3\sqrt{-1})(9 + 4\sqrt{-1})}{(9 - 4\sqrt{-1})(9 + 4\sqrt{-1})} \\ &= \frac{102 + 13\sqrt{-1}}{97} \\ &= 1\tfrac{5}{97} + \tfrac{13}{97}\sqrt{-1}. \end{aligned}$$

EXERCISE 89.

Perform the operations indicated:

1. $\sqrt{-9}+\sqrt{-81}$.
2. $\sqrt{-25}-2\sqrt{-4}$.
3. $\sqrt{-256}+3\sqrt{-81}-\sqrt{-144}$.
4. $\sqrt{-a^2}+2\sqrt{-a^2}$.
5. $\sqrt{-16x^2y^4}-2\sqrt{-4x^2y^4}$.
6. $\sqrt{-169m^2}+\sqrt{-225m^2}$.
7. $3a\sqrt{-9}+6a\sqrt{-4}-\sqrt{-81a^2}$.
8. $(3+2\sqrt{-1})\times(3-2\sqrt{-1})$.
9. $(5+\sqrt{-36})\times(5-6\sqrt{-1})$.
10. $(a+\sqrt{-b^2})\times(a-\sqrt{-b^2})$.
11. $(1+\sqrt{-1})\times(1-\sqrt{-1})$.
12. $(a\sqrt{-x^2y^4})\times(-b\sqrt{-x^2y^4})\times(c\sqrt{-x^2y^4})$.
13. $3\sqrt{-a^2}\times 4\sqrt{-b^2}\times -5\sqrt{-c^2}$.
14. $-\sqrt{-4}\times -\sqrt{-5}\times -\sqrt{-6}$.
15. $\sqrt{-14+2\sqrt{24}}$.
16. $\sqrt{-8+2\sqrt{12}}$.
17. $(\sqrt{-2}+\sqrt{-3}+\sqrt{-5})^2$.
18. $(\sqrt{-3}-\sqrt{-2})^3$.
19. $(\sqrt{-a^2}+2\sqrt{-b^2})^2$.
20. $(2+\sqrt{-1})\div(3+2\sqrt{-1})$.
21. $5+3\sqrt{-1}\div 7+2\sqrt{-1}$.
22. $11-\sqrt{-1}\div 13+\sqrt{-1}$.

CHAPTER XXV.

QUADRATIC EQUATIONS.

244. An equation of the second degree containing but one unknown quantity is called a **Quadratic Equation.**

245. A quadratic equation which involves only the square of the unknown quantity is called a **Pure Quadratic Equation.**

Thus, $x^2 = 9$, and $x^2 = 3a + b$.

246. A quadratic equation which involves both the square and the first power of the unknown quantity is called an **Affected Quadratic Equation.**

Thus, $x^2 + x = 12$, and $x^2 + x = a$.

PURE QUADRATIC EQUATIONS.

247. Every pure quadratic equation can be reduced to the form $x^2 = a$, and solved by extracting the square root of both members.

EXAMPLES:

(1) Solve $x^2 + 5 = 3x^2 - 3$.

Transposing, $-2x^2 = -8$.

Dividing by -2, $x^2 = 4$.

Extracting the square root of both members,

$$x = \pm 2.$$

248. The sign ± is used because the square root of a quantity is either positive or negative.

It might appear that the result should be $\pm x = \pm 2$, but as this result reduces to $x = \pm 2$, the two are coincident.

For, $\pm x = \pm 2$

gives

$$+x = +2, \quad (1)$$

$$+x = -2, \quad (2)$$

$$-x = -2, \quad (3)$$

$$-x = +2. \quad (4)$$

As (1) and (3) and (2) and (4) are identical, we have

$$x = \pm 2.$$

(2) Solve $x^2 - a + b = \frac{2x^2 + c - d}{3}$.

Clearing of fractions, $3x^2 - 3a + 3b = 2x^2 + c - d$.

Transposing and uniting, $x^2 = c - d + 3a - 3b$.

Hence, $x = \pm\sqrt{3a + c - 3b - d}$.

(3) Solve $x^2 + 5 = 3 - 2x^2$.

Transposing and uniting, $3x^2 = -2$.

Dividing by 3, $x^2 = -\frac{2}{3}$.

Extracting the square root, $x = \pm\sqrt{-\frac{2}{3}}$

$$= \pm\tfrac{1}{3}\sqrt{-6}.$$

Hence, the value of x is imaginary.

A **rational** root is one which can be found exactly.

A **surd** or **irrational** root is one that can be found approximately.

Rational and **Surd** roots are called **Real Roots.**

A root which can only be indicated is called an **Imaginary Root.**

EXERCISE 90.

Solve the following equations:

1. $x^2 - 4 = 0.$
2. $x^2 + 9 = 2x^2.$
3. $\frac{x^2}{3} = 2x^2 - 41\frac{2}{3}.$
4. $4x^2 - 1 = 0.$
5. $9x^2 = 16.$
6. $5x^2 = 2000.$
7. $3x^2 = 27a^2.$
8. $x^2 - 1 = a^2 - 1.$
9. $ax^2 = a^3.$
10. $x^2 = (a + b)^2.$
11. $2x^2 - a = 4x^2 - 48.$
12. $(a - 1)^2x^2 = a^2 - 1.$
13. $(a + 1)^3x^2 = a^3 + 1.$
14. $x^2 - 1 = (a + 1)^2.$
15. $\frac{x^2}{25} = \frac{1}{16}.$
16. $\frac{x^2 + 1}{x^2 - 2} = \frac{1}{2}.$
17. $\frac{(x + a - b)^2}{4} = \frac{a^2 + 2ab + b^2}{9}.$
18. $x^2 + \frac{1}{2} = a^2 - \frac{1}{2}.$
19. $\frac{2x^2}{4} - 3 + \frac{5x^2}{12} = \frac{14}{48} - x^2 + \frac{335}{24}.$

AFFECTED QUADRATIC EQUATIONS.

249. Every affected quadratic equation may be made to assume the form $x^2 \pm bx = c,$ (1) b and c being positive or negative. We may reduce to this form by the following process:

Place all terms containing the unknown quantity in the left-hand member.

Place all known terms in the right-hand member.

Divide by the coefficient of x^2. If x^2 is negative, change all signs.

To *complete the square* of the left-hand member of (1), we must add $\frac{b^2}{4}$.

Adding, therefore, $\frac{b^2}{4}$ to both members, we have,

$$x^2 + bx + \frac{b^2}{4} = c + \frac{b^2}{4}.$$

Extracting the square root,

$$x + \frac{b}{2} = \pm\sqrt{c + \frac{b^2}{4}}.$$

Solving these two simple equations, we have,

$$x = -\frac{b}{2} \pm \sqrt{c + \frac{b^2}{4}}$$
$$= -\tfrac{1}{2}(b \mp \sqrt{4c + b^2}).$$

250. Hence to solve an affected quadratic equation:

I. *Reduce the equation to the form* $x^2 + bx = c$.

II. *Complete the square by adding the square of half the coefficient of* x *to both sides.*

III. *Extract the square root.*

IV. *Reduce the resulting simple equations.*

251. In certain equations it is more convenient to solve by making the coefficient of x^2 a perfect square, and then completing the square.

252. The equation, when the coefficient of x^2 is a perfect square, assumes the form $a^2x^2 + bx = c$. To complete this square it is necessary to add $\frac{b^2}{4a^2}$ to both sides. That is, we add *the square of the coefficient of* x *divided by twice the square root of the coefficient of* x^2.

Examples:

(1) Solve $x^2 - 4x + 3 = 0$.

Transposing, $x^2 - 4x = -3$.

Completing the square, $x^2 - 4x + 4 = 1$.

Taking the square root, $x - 2 = \pm 1$.

Reducing, $x = +3$ or $+1$.

Clear of fractions and reduce to the form $x^2 + bx = c$.

(2) Solve $\dfrac{2}{x-1} - \dfrac{3}{x-2} = \dfrac{2}{x-4}$.

Clear of fractions,

$$2x^2 - 12x + 16 - 3x^2 + 15x - 12 = 2x^2 - 6x + 4.$$

Transposing and uniting, $-3x^2 + 9x = 0$.

Dividing by -3, $x^2 - 3x = 0$.

Completing the square, $x^2 - 3x + \frac{9}{4} = \frac{9}{4}$.

Taking the square root, $x - \frac{3}{2} = \pm\frac{3}{2}$.

Simplifying, $x = 3$ or 0.

These results may be verified by putting 3 or 0 for x in the original equation.

Make the coefficient of x^2 a perfect square.

(3) Solve $12x^2 + 10x + 2 = 1 + 5x$.

Transposing and uniting,

$$12x^2 + 5x = -1.$$

Making coefficient of x^2 a perfect square,

$$144x^2 + 60x = -12.$$

Completing the square,

$$144x^2 + 60x + (\tfrac{5}{2})^2 = -12 + (\tfrac{5}{2})^2$$
$$= \frac{-48 + 25}{4}.$$

Taking the square root, $12x + \frac{5}{2} = \pm\sqrt{-\frac{23}{4}}$

$$= \pm\tfrac{1}{2}\sqrt{-23}.$$

Reducing, $12x = -\frac{5}{2} \pm \frac{1}{2}\sqrt{-23}$.

$$\therefore\ x = -\tfrac{1}{24}(5 \mp \sqrt{-23}).$$

Showing that both values of x are imaginary.

Fractions may be avoided in completing the square by multiplying both members of the given equation by four times the coefficient of x^2.

SOLUTION OF LITERAL QUADRATICS.

Examples:

(1) Solve $x^2 - 2ax + b = 0$.

Transposing, $x^2 - 2ax = -b$.

Completing the square,

$$x^2 - 2ax + a^2 = a^2 - b.$$

Taking the square root, $x - a = \pm\sqrt{a^2 - b}$.

Reducing, $x = a + \sqrt{a^2 - b}$,

and $x = a - \sqrt{a^2 - b}$.

(2) Solve $(ax - b)(bx - a) = c^2$.

Simplify, $abx^2 - x(a^2 + b^2) = c^2 - ab$.

Multiply by $4ab$, $4a^2b^2x^2 - 4abx(a^2 + b^2) = 4abc^2 - 4a^2b^2$.

Complete the square,

$$4a^2b^2x^2 - 4abx(a^2 + b^2) + (a^2 + b^2)^2 = 4abc^2 - 4a^2b^2 + (a^2 + b^2)^2.$$

Take square root, $2abx - (a^2 + b^2) = \pm\sqrt{(a^2 - b^2)^2 + 4abc^2}$.

$$\therefore\ x = a^2 + b^2 \pm \sqrt{(a^2 - b^2)^2 + 4abc^2}.$$

(3) Solve $\dfrac{x^2}{(a+b)^2} + \dfrac{x}{a+b} = m$.

Multiply both members by 4,

$$\frac{4x^2}{(a+b)^2} + \frac{4x}{(a+b)} = 4m.$$

Complete the square,

$$\frac{4x^2}{(a+b)^2} + \frac{4x}{a+b} + 1 = 4m + 1.$$

Take the square root,

$$\frac{2x}{a+b} + 1 = \pm\sqrt{4m + 1}.$$

Reduce, $2x = -(a + b) \pm (a + b)\sqrt{4m + 1}$.

$$\therefore\ x = \tfrac{1}{2}\{-(a + b) \pm (a + b)\sqrt{4m + 1}\}.$$

(4) Solve $mx^2 - nx = p$.

Multiplying by $4m$, we have,

$$4m^2x^2 - 4mnx = 4mp.$$

Completing the square,

$$4m^2x^2 - 4mnx + n^2 = 4mp + n^2.$$

Taking the square root,

$$2mx - n = \pm\sqrt{4mp + n^2}.$$

Reducing,
$$x = \frac{n \pm \sqrt{4mp + n^2}}{2m}.$$

EXERCISE 91.

Solve the following equations:

1. $7x^2 + 12 = 16x - 3.$
2. $5x^2 - 17x = -12.$
3. $3x^2 - 5x = -2.$
4. $9x^2 - 2 = 15x + 4.$
5. $x^2 + 11 = 6x + 10.$
6. $x^2 + 15 = x + 16.$
7. $3x^2 + 10x = 6\frac{1}{3}.$
8. $5x^2 - 2x + \frac{1}{2} = \frac{27}{10}.$
9. $\dfrac{2}{3} + \dfrac{2}{3 + x} = \dfrac{2}{3 - 2x}.$
10. $\dfrac{x + 1}{x + 2} - \dfrac{x - 2}{x + 1} = \dfrac{3}{4}.$
11. $\dfrac{\frac{2}{3}}{\frac{2}{3} + x} - \dfrac{2}{3} = \dfrac{2}{3} - \dfrac{\frac{2}{3}x + \frac{2}{3}}{\frac{2}{3} + x}.$
12. $\dfrac{4x + 1}{4x - 1} + \dfrac{5x - 5}{6x - 2} = 1\frac{2}{3}.$
13. $(x - 1)(x + 2) - (x + 3)(x - 4) = 20.$
14. $\frac{1}{4}(x - \frac{1}{2})(x - \frac{1}{3}) - \frac{1}{2}(x - \frac{1}{6})(x + \frac{1}{2}) = -\frac{29}{72}.$
15. $\dfrac{1}{x - 1} + \dfrac{1}{x + 2} + \dfrac{1}{x - 3} = \dfrac{7}{10}.$

16. $\dfrac{1}{x+1}+\dfrac{1}{x-1}+\dfrac{2}{x^2-1}=\dfrac{3}{5}.$

17. $2ax^2+4bx=8c.$

18. $ax^2+bx=a+1.$

19. $x^2(a+b)+x(a-b)=c.$

20. $3x^2=18ax-15a^2.$

21. $\dfrac{2x^2}{a+b}+2a=\dfrac{4ax}{a+b}+2b.$

22. $4ax^2-4x(a-1)=44.$

23. $\dfrac{1}{a-b}+\dfrac{2}{x}=\dfrac{x(b-a)+2ab}{abx}.$

24. $ax(x-1)+b(x+1)=10b.$

25. $x^2-(c-b)x=(a-b)(a-c).$

26. $x^2-2a^2x-2b^2x=a^4-2a^2b^2+b^4.$

27. $ax^2-a^2x-x=-a.$

SOLUTIONS BY FORMULA AND FACTORING.

253. When b and c are positive or negative, every affected quadratic, as has been shown, may be reduced to the form $x^2+bx=c$.

Solving this equation, we have,

$$x^2+bx+\frac{b^2}{4}=c+\frac{b^2}{4},$$

$$x+\frac{b}{2}=\pm\sqrt{c+\frac{b^2}{4}}.$$

$$\therefore\ x=-\frac{b}{2}\pm\sqrt{c+\frac{b^2}{4}}.$$

Now, if in any given equation of the form $x^2+bx=c$, we substitute the values of b and c in this formula,

$$-\frac{b}{2}\pm\sqrt{c+\frac{b^2}{4}},$$

we shall have both values of x.

Write the answer of the following equation by substituting the values of b and c in the formula

$$3x^2 + 8x = 115.$$

Make the equation of the form $x^2 + bx = c$ by dividing by 3.

$$x^2 + \tfrac{8}{3}x = 38\tfrac{1}{3}.$$

Here, $b = \tfrac{8}{3},\ c = 38\tfrac{1}{3}.$

Hence,
$$-\frac{b}{2} \pm \sqrt{c + \frac{b^2}{4}} = -\frac{4}{3} \pm \sqrt{38\tfrac{1}{3} + \tfrac{16}{9}}$$
$$= -\tfrac{4}{3} \pm \sqrt{\tfrac{361}{9}}.$$
$$\therefore\ x = -\tfrac{4}{3} + \tfrac{19}{3} = \tfrac{15}{3} = 3;$$

or,
$$= -\tfrac{4}{3} - \tfrac{19}{3} = -\tfrac{23}{3} = -7\tfrac{2}{3}.$$

254. An affected quadratic equation, reduced to the form $x^2 + bx - c = 0$, may often be solved by inspection, by resolving the left-hand member into factors.

EXAMPLES:

(1) Solve $x^2 + 3x + 2 = 0$.

Factoring, $(x + 2)(x + 1) = 0.$

Dividing by $(x + 2)$, $(x + 1) = 0.$

$$\therefore\ x = -1.$$

Dividing by $(x + 1)$, $(x + 2) = 0.$

$$\therefore\ x = -2.$$

Hence the roots -2 and -1.

This method may be applied to higher equations.

(2) Solve $x^3 - ax^2 - x^2 - 2x + ax + 2a = 0$.

Factoring,
$$x^2(x - a) - x(x - a) - 2(x - a) = 0$$
$$= (x - a)(x^2 - x - 2) = 0$$
$$= (x - a)(x - 2)(x + 1) = 0.$$

Hence,
$$\begin{cases} x = a, \\ x = 2, \\ x = -1. \end{cases}$$

(3) Solve $x(x+1)(x+2) = m(m+1)(m+2)$.

Reducing, we have, $x^3 + 3x^2 + 2x - m^3 - 3m^2 - 2m = 0$.

Factoring, $(x^3 - m^3) + (3x^2 - 3m^2) + (2x - 2m) = 0$

$$= (x-m)(x^2 + xm + m^2) + 3(x-m)(x+m) + 2(x-m) = 0$$

$$= (x-m)(x^2 + xm + m^2 + 3x + 3m + 2) = 0.$$

$$\therefore\ x = m.$$

Solving $x^2 + xm + 3x = -m^2 - 3m - 2$, we find,

$$x = -\frac{m+3}{2} \pm \frac{1}{2}\sqrt{1 - 6a + 3a^2}.$$

EXERCISE 92.

Find the roots of the following equations:

1. $x^2 + 7x + 12 = 0$.
2. $x^2 - 7x + 12 = 0$.
3. $2x^2 + 3x + 1 = 0$.
4. $(x+3)(x-2)(x^2 - x - 2) = 0$.
5. $12x^2 - 5x - 2 = 0$.
6. $6x^2 - 19x + 10 = 0$.
7. $4x^2 + 11x - 3 = 0$.
8. $(x-a)(x-b)(x^2 - 2ax + a^2) = 0$.
9. $x(x^2-1)(x^2-a^2) = 0$.
10. $x^3 - 1 = 0$.
11. $x^6 - 27a^3 = 0$.
12. $125x^3 - a^3 = 0$.
13. $x^4 - 16 - (x^2 - 4) = 0$.
14. $(x-1)(x-2)(x^2 - 12x + 20) = 0$.
15. $(n^3 - 1) + (n^2 - 1) - (n - 1) = 0$.

EQUATIONS IN QUADRATIC FORM, AND QUADRATICS INVOLVING RADICALS.

255. Equations containing two powers of the unknown quantity, the exponent of one being twice that of the other, may be solved after the manner of quadratic equations.

EXAMPLES:

(1) Solve $x^6 - 35x^3 + 216 = 0$.

Transpose, $x^6 - 35x^3 = -216$.

Multiply by 4, $4x^6 - 140x^3 = -864$.

Complete square, $4x^6 - 140x^3 + (35)^2 = -864 + (35)^2$,

$$4x^6 - 140x^3 + (35)^2 = 361.$$

Take square root, $2x^3 - 35 = \pm 19$,

and $2x^3 = 54$ or 16.

$\therefore\ x^3 = 27$ or 8.

From which we have, $x = 3$ or 2.

To find the remaining root we must proceed as follows:

$$x^3 - 27 = 0.$$

Hence, $(x-3)(x^2 + 3x + 9) = 0$.

$\therefore\ x = 3$.

Solve $x^2 + 3x + 9 = 0$.

$$x^2 + 3x = -9,$$
$$x^2 + 3x + \tfrac{9}{4} = -9 + \tfrac{9}{4},$$
$$x^2 + 3x + \tfrac{9}{4} = -\tfrac{27}{4},$$
$$x + \tfrac{3}{2} = \pm\sqrt{-\tfrac{27}{4}},$$
$$x = -\tfrac{3}{2} \pm \tfrac{3}{2}\sqrt{-3} = -\tfrac{3}{2}(1 \mp \sqrt{-3}).$$

Again, $x - 8 = 0$.

Hence, $(x-2)(x^2 + 2x + 4) = 0$.

$\therefore\ x = 2$.

Solve $x^2 + 2x + 4 = 0$.

$$x^2 + 2x = -4,$$
$$x^2 + 2x + 1 = -3,$$
$$x + 1 = \pm\sqrt{-3},$$
$$x = -1 \pm \sqrt{-3}.$$

$$\therefore\ x = 2,\ 3,\ -1 \pm \sqrt{-3},\ -\tfrac{3}{2}(1 \mp \sqrt{-3}).$$

(2) Solve $x^4 - x^3 + x^2 - x = 0$.

Factor, $$x^3(x-1) + x(x-1) = 0,$$
$$x(x^2+1)(x-1) = 0.$$

Hence, $$\begin{cases} x = 0, \\ x = 1, \\ x = \pm\sqrt{-1}. \end{cases}$$

256. In equations involving radical expressions we must free the equations of radicals, and proceed as in simple quadratics.

EXAMPLES:

(1) Solve $7x - 2\sqrt{x} = 2$.

Transpose $7x$ and square, $4x = 4 - 28x + 49x^2$.

Arrange, $49x^2 - 32x = -4$.

Complete square,
$$49x^2 - 32x + (\tfrac{16}{7})^2 = -4 + \tfrac{256}{49} = \tfrac{60}{49}.$$

Take square root, $7x - \frac{16}{7} = \pm\frac{2}{7}\sqrt{15}$.
$$\therefore\ x = \tfrac{2}{49}(8 \pm \sqrt{15}).$$

257. Some radical equations may be solved without clearing of radicals in the first instance.

In the equation $x^2 + 5x + \sqrt{x^2+5x+1} = 29$ it is readily seen that if 1 be added to both members, the equation will be of the form $x^2 + bx = c$, since $x^2 + 5x + 1$ is the square of $\sqrt{x^2+5x+1}$.

(2) Solving, we have, by adding 1 to each side,
$$x^2 + 5x + 1 + \sqrt{x^2+5x+1} = 30.$$

Completing square, $x^2 + 5x + 1 + \sqrt{x^2+5x+1} + \frac{1}{4} = \frac{121}{4}$.

Taking square root, $\sqrt{x^2+5x+1} + \frac{1}{2} = \pm\frac{11}{2}$,

and $\sqrt{x^2+5x+1} = \frac{10}{2}$ or $-\frac{12}{2}, = 5$ or -6.

Squaring this last, $x^2 + 5x + 1 = 25$ or 36.

Solving these,
$$x^2 + 5x = 24,$$
$$x^2 + 5x + (\tfrac{5}{2})^2 = 24 + (\tfrac{5}{2})^2,$$
$$x^2 + 5x + (\tfrac{5}{2})^2 = \tfrac{121}{4},$$
$$x + \tfrac{5}{2} = \pm \tfrac{11}{2}.$$
$$\therefore\ x = 3 \text{ or } -8.$$

Again,
$$x^2 + 5x + 1 = 36.$$
$$x^2 + 5x = 35.$$
$$x^2 + 5x + (\tfrac{5}{2})^2 = 35 + (\tfrac{5}{2})^2.$$
$$x + \tfrac{5}{2} = \pm \tfrac{1}{2}\sqrt{165}.$$
$$\therefore\ x = -\tfrac{5}{2} \pm \tfrac{1}{2}\sqrt{165}.$$
$$= -\tfrac{1}{2}(5 \mp \sqrt{165}).$$

The device of letting $x^2 + 5x + 1 = y^2$ and $\sqrt{x^2 + 5x + 1} = y$ is sometimes employed in solving the above equation.

From such substitution, we have,
$$y^2 + y = 30.$$

And solving, we have,
$$y = +5 \text{ or } -6.$$

Now, substituting for each of these values, we have,
$$\sqrt{x^2 + 5x + 1} = 5 \text{ or } -6.$$

Solve as before, and obtain four values of x.

EXERCISE 93.

Solve the following equations:

1. $8x^6 - 9x^3 + 5 = 4$.
2. $27x^6 - 217x^3 + 8 = 0$.
3. $6x^6 + 48 = 54x^3$.
4. $2x^4 + 288 = 50x^2$.
5. $x^{10} + 31x^5 = 32$.
6. $x^3 - x^{\frac{3}{2}} - 56 = 0$.
7. $x^{-2n} - 2x^{-n} = 8$.
8. $x^{\frac{4}{9}} - x^{\frac{2}{9}} - 56 = 0$.
9. $\dfrac{1}{x^4} + \dfrac{1}{x^2} = \dfrac{1}{8}$.
10. $(x^2 + 1)^2 + (x^2 + 1) - 28 = 0$.
11. $\frac{1}{2}(x^2 + a)^4 - \frac{1}{4}(x^2 + a)^2 = 4c^2$.

12. $x^9 + 21x = -22x^5.$

13. $x^7 = 4x^4 + 621x.$

14. $(x^2 - 1)^2 - 3(x^2 - 1) = 40.$

15. $(3x^2 + 1)^{\frac{1}{2}} + \frac{1}{2}(3x^2 + 1)^{\frac{1}{4}} = \frac{1}{2}(4 + \sqrt{2}).$

16. $\sqrt{x - 1} + \sqrt{x + 1} = 2 + \sqrt{2}.$

17. $x^2 + \frac{1}{x^2} + \frac{1}{6}\left(x - \frac{1}{x}\right) = \frac{86}{9}.$

18. $3x^2 + 7x + 6\sqrt{3x^2 + 7x + 1} = 54.$

19. $2(x^2 + x - 1) - 6\sqrt{x^2 + x - 1} = 0.$

20. $\sqrt{10 + x} - \sqrt[4]{10 + x} - 2 = 0.$

21. $\sqrt{x + a} + \sqrt{x - a} = bc.$

22. $x^{4m} + 4x^{2m} + 3 = 0.$

23. $4x^{\frac{1}{2m}} + 9x^{\frac{1}{4m}} + 2 = 0.$

24. $x^2 - 2x\sqrt{-1} = -1.$

25. $x^2 + a^2 - b^2 = c^2 + 2ax - 2bc.$

26. $\frac{x^2}{(a + b)^2} - \frac{4ab}{(a + b)^2}x - (a - b)^2 = 0.$

27. $x^2 - (m^2 + n^2)x = -mn(m^2 - n^2).$

28. $\frac{x + m}{x - m} - \frac{x - m}{x + m} = \frac{4m(m + n)}{n(2n + m)}.$

29. $\frac{2 + 3x}{1 - 4x} - \frac{6 - 5x}{7x - 25} = \frac{8\sqrt{4} - x}{28x - 193}.$

30. $\frac{\sqrt{3a + x}}{\sqrt{a} + \sqrt{3a + x}} - \frac{\sqrt{3a - x}}{\sqrt{a} - \sqrt{3a - x}} = 0.$

31. $\frac{x + m}{a + m} - \frac{a - m}{x - m} = \frac{x + n}{a + n} - \frac{a - n}{x - n}.$

32. $169x^4 + 144x^2 - 25 = 0.$

33. $\sqrt{2x+2}+\sqrt{7x+6}=\sqrt{7x+72}.$

34. $\sqrt{x^2-8x+31}+x^2-8x+16=5.$

35. $\sqrt{x}+2\sqrt[4]{x}=8.$

36. $\sqrt{x}\sqrt[4]{x^2-1}-2x\sqrt{x^2-1}=\frac{1}{4}.$

37. $\sqrt[3]{(a+x)^2}+6\sqrt[3]{(a-x)^2}=5\sqrt[3]{a^2-x^2}.$

38. $12x^{-\frac{3}{4}}-x^{-\frac{3}{8}}=\frac{1}{16}.$

39. $\sqrt[3]{x+\sqrt{2}}-\sqrt[3]{x-\sqrt{2}}=\sqrt{2}.$

40. $\sqrt[3]{x^2+2x\sqrt{m}+m}-\sqrt[3]{x^2+2x\sqrt{m}-m}=\sqrt[3]{2m}.$

PROBLEMS INVOLVING QUADRATICS.

(1) Solve:

If a certain integral number is divided into 12, and the quotient is added to the number, the sum will be $11\frac{1}{5}$.

Let $x=$ the number,

and $\frac{12}{x}=$ 12 divided by the number.

Then, $x+\frac{12}{x}=11\frac{1}{5}.$

Clearing of fractions, $5x^2+60=56x.$

Transposing, $5x^2-56x=-60.$

Multiplying by 20, $100x^2-1120x=-1200.$

Completing the square,

$$100x^2-1120x+(56)^2=1936.$$

Taking square root, $10x-56=\pm 44.$

Reducing, $10x=100 \text{ or } 12.$

$\therefore\ x=10 \text{ or } 1\frac{1}{5}.$

258. The answers, therefore, are 10 and $1\frac{1}{5}$. Now, 10 will satisfy the conditions of the problem, but $1\frac{1}{5}$ will not satisfy these conditions. Hence, in the solution of problems involving quadratics, when two or more answers are obtainable, only those answers which are applicable and satisfy the conditions of the problem are taken.

If the roots are imaginary, the problem is impossible.

EXERCISE 94.

1. The sum of the squares of two consecutive numbers is 61. What are the numbers?

2. The product of two consecutive numbers is equal to $\frac{3}{4}$ the square of the second number. What are the numbers?

3. If I multiply the number of dollars in my pocket by itself, I shall have $ 132.25. How many dollars have I?

4. The product of two numbers is 7007, and the numbers themselves are to each other as 11 to 13. What are the numbers?

5. Find two numbers which are to each other as $\frac{1}{2} : \frac{1}{4}$, and the sum of whose squares is 3125.

6. If I add 15 to one side of a square and subtract 2 from an adjacent side, I shall have a rectangle whose area is twice that of the square. Find the side of the square.

7. A person buys a certain number of pineapples, and pays as many cents for each pineapple as there are pineapples. For all he pays $6\frac{1}{4}$ dollars. How many did he buy?

8. A person buys a certain number of pounds of starch, four times as many of sugar, and eight times as many of coffee. He pays for a pound of each of the three forty times as many cents as the number of pounds of each purchased. Altogether he pays $32\frac{2}{5}$ dollars. How many pounds of each does he buy?

9. A rectangular garden is 37 yards wide, and 259 yards long. The width will be increased by a certain number of yards, and the length diminished by seven times as many, if

63 square yards are taken from the whole. What is the number of increase in width?

10. If I increase a number by 3, and also diminish it by 3, the sum of the quotients obtained by the division of the larger number into the smaller, and the smaller into the larger, is $\frac{10}{3}$. What is the number?

11. Divide the number 20 into two parts so that the squares may be in the ratio of 4 to 9.

12. If I increase the side of a square by 3 yards, making its area $\frac{9}{16}$ larger, what is the length of the original side?

13. Two men start at the same time from the same place, and go in directions at right angles to each other. The first goes $4\frac{1}{2}$ miles per hour, the other 6 miles per hour. After how many hours will they be 30 miles distant from one another?

14. The square of a certain number added to 12 times the number is equal to the cube of the number itself. What is the number?

15. What number is it whose cube is equal to the square of its double?

16. What number added to its reciprocal gives 4.25?

17. What number subtracted from its reciprocal gives 6.09?

CHAPTER XXVI.

PROPERTIES OF QUADRATICS.

259. *A quadratic equation has two, and only two, roots.*

Consider the equation $x^2+px=q$.

Suppose this equation to have three different roots, r_1, r_2, and r_3; that is, three different values for x.

Then we should have,

$$r_1^2+pr_1=q. \quad (1)$$

$$r_2^2+pr_2=q. \quad (2)$$

$$r_3^2+pr_3=q. \quad (3)$$

Subtracting (2) from (1), we have,

$$r_1^2-r_2^2+pr_1-pr_2=0 \quad (4)$$

$$=(r_1-r_2)(r_1+r_2)+p(r_1-r_2)=0 \quad (5)$$

$$=(r_1-r_2)(r_1+r_2+p)=0. \quad (6)$$

Hence, $r_1-r_2=0$, or $r_1+r_2+p=0$; but as the roots were different, by hypothesis, r_1-r_2 cannot equal zero.

Hence, $$r_1+r_2+p=0. \quad (7)$$

In like manner, from (3) and (1), we have,

$$r_1+r_3+p=0. \quad (8)$$

Now, subtracting (7) from (8), we have, $r_2-r_3=0$, which is impossible.

Hence, a quadratic equation cannot have more than two roots.

Again, consider the equation $x^2+px=q$. Let r_1 and r_2 be its roots.

Solving, we have,

$$r_1 = -\frac{p}{2} + \sqrt{\frac{p^2 + 4q}{4}}, \qquad (1)$$

and

$$r_2 = -\frac{p}{2} - \sqrt{\frac{p^2 + 4q}{4}}. \qquad (2)$$

Adding (1) and (2), we have,

$$r_1 + r_2 = -p.$$

Multiplying (1) by (2), we have,

$$r_1 r_2 = -q.$$

260. *Hence, if a quadratic equation is of the form $x^2 + px = q$, the sum of the roots is equal to the coefficient of x with opposite sign, and the product of the roots is equal to the second member with opposite sign.*

Again, substituting the values of p and q in the equation $x^2 + px = q$, we have, $x_2 - (r_1 + r_2)x + r_1 r_2 = 0$. Factoring which, we have,

$$(x - r_1)(x - r_2) = 0.$$

That is, any quadratic equation of the form $x^2 + px = q$ may be written,

$$(x - r_1)(x - r_2) = 0.$$

261. *Hence, when the roots of a quadratic equation are given, the equation may be formed by subtracting each root from x, and placing the product of the resulting expressions equal to zero.*

CHARACTER OF THE ROOTS OF A QUADRATIC EQUATION.

The general quadratic equation $ax^2 + bx + c = 0$ gives the roots,

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a},$$

$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

It is evident that,

I. *If* $b^2 - 4ac$ *is positive and greater than zero, the roots are real and unequal.*

II. *If* $b^2 - 4ac$ *equals zero, the roots are real and equal.*

III. *If* $b^2 - 4ac$ *is less than zero, the roots are imaginary.*

EXAMPLES:

(1) Find the sum and the product of the roots of

$$3x^2 - 7x + 1 = 0.$$

Transpose, $3x^2 - 7x = -1.$

Divide by 3, $x^2 - \frac{7}{3}x = -\frac{1}{3}.$

Hence the sum of the roots is equal to $\frac{7}{3}$, and the product is equal to $\frac{1}{3}$.

EXERCISE 95.

Find by inspection the sum and the product of the roots of the following equations:

1. $x^2 + 12x + 7 = 0.$
2. $3x^2 - 4x - 1 = 0.$
3. $7x^2 - 2x + 3 = 0.$
4. $4x^2 + x + 1 = 0.$
5. $8x^2 - 9x - 9 = 0.$
6. $7x^2 - 8x + 6 = 0.$
7. $x^2 - 4ax + 4a^2 - 4b^2 = 0.$
8. $x^2 - a^2 - x - a = 0.$
9. $8x^2 - ax - 4a^2 = 0.$
10. $(m+1)^2x^2 + (m+1)x + (m+1)^{-2} = 0.$
11. $(a+b)x^2 + (a-b)x = -(a^2+b^2).$
12. $x^{-4} - x^{-2} - 1 = 0.$
13. $3x^{2n} + 6x^n + 1 = 0.$

(2) Find the quadratic equation whose roots are $-\frac{3}{4}$ and $\frac{7}{3}$

By (261) we have $(x + \frac{3}{4})(x - \frac{7}{3}) = 0.$

By multiplication, $x^2 - \frac{19}{12}x - \frac{7}{4} = 0.$

Hence, $12x^2 - 19x - 21 = 0.$

EXERCISE 96.

Form quadratic equations having the following roots:

1. $3, 4.$
2. $6, \frac{1}{2}.$
3. $7, \frac{3}{4}.$
4. $9, \frac{5}{7}.$
5. $-\frac{1}{2}, +\frac{3}{4}.$
6. $(a-b), (a+b).$
7. $-(m+n), -m.$
8. $\frac{1}{2}\sqrt{2}, -\frac{1}{3}\sqrt{3}.$
9. $(a+1), (\sqrt{a}-1).$
10. $\sqrt{a}, -\sqrt{b}.$
11. $3a\sqrt{\frac{1}{2}}, -2a\sqrt{b}.$
12. $a^2 - b\sqrt{a}, a^2 + b\sqrt{a}.$
13. $\frac{a}{\sqrt{b}}, -\frac{a}{\sqrt{c}}.$
14. $2+5\sqrt{2}, 2-5\sqrt{2}.$
15. $\sqrt{a+b}, -\sqrt{a^2+1}.$
16. $\frac{m-\sqrt{2}}{3}, \frac{m+\sqrt{2}}{3}.$
17. $a^{-2}, b^{-\frac{3}{4}}.$

(3) Determine the nature of the roots of the quadratic equations:

$$x^2 + 5x + 3 = 0.$$

Here, $a = 1,\ b = 5,\ c = 3.$

Hence, $b^2 - 4ac = 25 - 12 = 13.$

Therefore, the roots are real and unequal.

(4) $$16x^2 - 8x + 1 = 0.$$

Here, $a = 16,\ b = -8,\ c = 1.$

Hence, $b^2 - 4ac = 64 - 64 = 0.$

Therefore, the roots are real and equal.

(5) $$3x^2 - 2x + 4.$$

Here, $a = 3,\ b = -2,\ c = 4.$

Hence, $b^2 - 4ac = 4 - 48 = -44.$

Therefore, the roots are imaginary.

(6) Find the values for m, for which the following equation has two equal roots:

$$(m-2)x^2 + (m-5)x + 2m - 5 = 0.$$

Here, $a = m - 2,\ b = m - 5,\ c = 2m - 5.$

Now, if the roots are to be equal,

$$b^2 - 4ac = (m-5)^2 - 4(m-2)(2m-5) = 0$$
$$= 7m^2 - 26m + 15 = 0.$$

Solving, we have, $m = \frac{5}{7}$ or 3.

Substituting these values in the given equation, we have,

$$x^2 - 2x + 1 = 0,$$

and $$9x^2 + 30x + 25 = 0,$$

which equations have equal roots.

EXERCISE 97.

Determine without solving the character of the root of the following quadratic equations:

1. $x^2 - 9x + 20 = 0.$
2. $x^2 + 11x + 30 = 0.$
3. $x^2 - 3x - 4 = 0.$
4. $2x^2 + 3x + 1 = 0.$
5. $16x^2 - 8x + 1 = 0.$
6. $21x^2 + 47x + 20 = 0.$
7. $x^2 + 8x + 16 = 0.$
8. $7x^2 + 6x - 1 = 0.$
9. $6x - 5x + 4 = 0.$
10. $7x - 9x + 11 = 0.$
11. $8x + 65x + 8 = 0.$
12. $100x^2 + 20x + 1 = 0.$

EXERCISE 98.

Determine the values for m in the following quadratic equations which shall give equal values for x:

1. $(3m+1)x^2 + 2mx + 2x + m = 0.$
2. $(m+1)x^2 + (m-1)x + 2m + 2 = 0.$
3. $(m-3)x^2 + (m+4)x + 6m = 0.$
4. $2mx^2 - 2x^2 - 3mx + 6x + 3m = 0.$
5. $(m-2)x^2 + (m+2)x - 3m = 0.$

RESOLUTION OF QUADRATICS INTO FACTORS.

262. The principles already set forth in Article (259) serve for the resolution of quadratic expressions of the form $ax + bx + c$ into factors, each of which shall contain the first degree of x only.

The equation $ax^2 + bx + c = 0$ may be written

$$a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = 0, \text{ and } \left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = 0.$$

Letting r_1 and r_2 represent the roots, we have

$$(x - r_1)(x - r_2) = 0,$$

and $$\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = (x - r_1)(x - r_2).$$

Therefore, $$ax^2 + bx + c = a(x - r_1)(x - r_2).$$

Factor, by applying the foregoing principles, the expression $2x^2 + 7x - 15 = 0$. Placing the expression equal to zero, we have, $2x^2 + 7x - 15 = 0$;

or, $$2\left(x^2 + \tfrac{7}{2}x - \tfrac{15}{2}\right) = 0, \text{ and } x^2 + \tfrac{7}{2}x - \tfrac{15}{2} = 0.$$

Now, $$x^2 + \tfrac{7}{2}x - \tfrac{15}{2} = (x - r_1)(x - r_2).$$

$$\therefore\ x = -\frac{7}{4} \pm \sqrt{\frac{\frac{49}{4} + 30}{4}} = \frac{6}{4}, \text{ or } -\frac{20}{4} = \frac{3}{2} \text{ or } -5.$$

Here, $$a = 2,\ r_1 = \tfrac{3}{2},\ r_2 = -5.$$

$$\therefore\ 2x^2 + 7x - 15 = (2x - 3)(x + 5).$$

EXERCISE 99.

Factor the quadratic expressions by the principles set forth:

1. $x^2 - 15x + 56 = 0$.
2. $x^2 + 17x + 72 = 0$.
3. $x^2 - x - 56 = 0$.
4. $17x^2 + 18x + 1 = 0$.
5. $8x^2 + 18x + 9 = 0$.
6. $25x^2 - 10x + 1 = 0$.
7. $8 - 6x + x^2 = 0$.
8. $24x^2 - 25ax + a^2 = 0$.
9. $16x^2 - 24ax + 9a^2 = 0$.
10. $33x^2 + 130mx + 33m^2 = 0$.
11. $72x^2 + 58ax + 10a^2 = 0$.

CHAPTER XXVII.

SIMULTANEOUS EQUATIONS INVOLVING QUADRATICS.

263. Various methods are employed for solving simultaneous equations where one or more may be of a degree higher than the first. The student must learn the proper application of the methods by experience. In no other section of Algebra can a student's ingenuity be so well exercised as in the solution of simultaneous quadratics.

The expressions $x = \pm 5, y = \pm 6,$
may be read $x = +5, y = +6,$
or, $x = -5, y = -6;$
and the expressions $x = \pm 5, y = \mp 6,$
may be read $x = +5, y = -6,$
or, $x = -5, y = +6.$

We shall now explain by means of examples the different cases of solution of simultaneous quadratics.

264. CASE I. — SUBSTITUTION. *That is, the value of one unknown quantity is substituted in terms of the other.*

265. CASE II. — ADDITION AND SUBTRACTION. *That is, one unknown quantity is eliminated by addition or subtraction, either in the first instance, or after some simple operation has been performed.*

266. CASE III. — SUBSTITUTING nx FOR y. *When the equations are of the second degree and homogeneous.*

267. Case IV.—Removing the Highest Powers of x and y. *When the equations are symmetrical with respect to x and y.*

268. Case V.—Substituting $a+b$ for x and $a-b$ for y. *When the equations are symmetrical with respect to x and y.*

269. Case VI.—Division and Factoring. *When the equations admit of factoring or the exact division of one into the other.*

Case I.—Solve the equations,

$$x+2y=8, \tag{1}$$

$$2x^2+y^2=17. \tag{2}$$

In (1)

$$x=8-2y.$$

$$\therefore\ x^2=64-32y+4y^2.$$

Substitute this value of x^2 in (2), and we have,

$$2(64-32y+4y^2)+y^2=17. \tag{3}$$

Simplify,

$$9y^2-64y=-111.$$

Completing the square, we have,

$$9y^2-64y+(\tfrac{32}{3})^2=-111+(\tfrac{32}{3})^2.$$

Taking the square root of each member, we have,

$$3y-\tfrac{32}{3}=\pm\tfrac{5}{3}.$$

Reducing,

$$y=\tfrac{37}{9} \text{ or } 3.$$

Substitute these values of y in (1), and we have,

$$x+6=8.$$

$$\therefore\ x=2.$$

And

$$x+\tfrac{74}{9}=8.$$

$$\therefore\ x=-\tfrac{2}{9}.$$

Therefore, if $x=2,\ y=3,$

and if $x=-\tfrac{2}{9},\ y=\tfrac{37}{9}.$

Case II. — Solve the equations,

$$x^2 + xy + y^2 = 91, \qquad (1)$$

$$x^2 - xy + y^2 = 31. \qquad (2)$$

Subtracting (2) from (1),

$$2xy = 60,$$

and $$xy = 30. \qquad (3)$$

Adding (3) to (1), we have,

$$x^2 + 2xy + y^2 = 121,$$

and $$x + y = \pm 11. \qquad (4)$$

Subtracting (3) from (2), we have,

$$x^2 - 2xy + y^2 = 1,$$

and $$x - y = \pm 1. \qquad (5)$$

From (4) and (5),

$$2x = +12, \ +10, \ -12, \ -10.$$

$$\therefore x = \ 6, \quad 5, \ -\ 6, \text{ and } -5.$$

Subtracting (5) from (4).

$$2y = 10, \ -10, \ +12, \text{ and } -12.$$

$$\therefore y = \ 5, \ -\ 5, \ +\ 6, \text{ and } -\ 6.$$

As the equations in the above examples are of the second degree and homogeneous, a solution may also be had by *Case III.*

Case III. — Solve the equations:

$$2x^2 - xy = 6, \qquad (1)$$

$$x^2 + y^2 = 25. \qquad (2)$$

Substituting nx for y, we have,

$$2x^2 - nx^2 = 6, \qquad (3)$$

$$x^2 + n^2x^2 = 25. \qquad (4)$$

From (3), $$x^2 = \frac{6}{2-n}.$$

From (4), $$x^2 = \frac{25}{1+n^2}.$$

Hence, $$\frac{6}{2-n} = \frac{25}{1+n^2},$$

and $$6 + 6n^2 = 50 - 25n.$$

Reducing, $$6n^2 + 25n = 44.$$

Multiplying by 6, $$36n^2 + 150n = 264.$$

Completing the square,

$$36n^2 + 150n + (\tfrac{25}{2})^2 = 264 + (\tfrac{25}{2})^2.$$

and $$6n + \tfrac{25}{2} = \pm\tfrac{41}{2}.$$

$$\therefore\ n = \tfrac{4}{3}, \text{ or } -\tfrac{11}{2}.$$

Substituting $\frac{4}{3}$ in (4) for n, we have,

$$x^2 + (\tfrac{4}{3})^2x^2 = 25,$$

and $$9x^2 + 16x^2 = 225.$$

Uniting, $$25x^2 = 225,$$

and $$x^2 = 9.$$

$$\therefore\ x = \pm 3.$$

Substituting these values of x in (2), we have,

$$y^2 = 16.$$

$$\therefore\ y = \pm 4.$$

Substituting $-\frac{11}{2}$ in (4), we obtain other values of x, and substituting these values in (1) or (2), we obtain other values of y.

Case IV.—Solve the equations,

$$x + y = 3, \qquad (1)$$

$$x^4 + y^4 = 17. \qquad (2)$$

Raising (1) to the fourth power, we have,

$$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 = 81 \qquad (3)$$

Adding (2), $$x^4 \qquad\qquad\qquad\qquad + y^4 = 17$$

and (3), $$2x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 2y^4 = 98 \qquad (4)$$

Dividing (4) by 2 and taking the square root of the quotient, we have,
$$x^2+xy+y^2=\pm 7. \qquad (5)$$

Squaring (1), $$x^2+2xy+y^2=9. \qquad (6)$$

Subtracting (5) from (6), $$xy=+16 \text{ or } +2. \qquad (7)$$

Solving
$$\left.\begin{aligned}x+y&=3\\xy&=2\end{aligned}\right\},\text{ and }\left.\begin{aligned}x+y&=3\\xy&=16\end{aligned}\right\},$$

we have,
$$\left.\begin{aligned}x&=2\\y&=1\end{aligned}\right\},\text{ or }\left.\begin{aligned}x&=1\\y&=2\end{aligned}\right\};$$

and
$$x=\frac{3\pm\sqrt{-55}}{2},$$
$$y=\frac{3\mp\sqrt{-55}}{2}.$$

Case V. — Solve the equations:
$$x^5+y^5=275, \qquad (1)$$
$$x+y=5. \qquad (2)$$

Putting $x=a+b$ and $y=a-b$, we have, from (1),
$$2a^5+20a^3b^2+10ab^4=275, \qquad (3)$$

and, from (2), $$2a=5. \qquad (4)$$

Dividing (3) by (4),
$$a^4+10a^2b^2+5b^4=55. \qquad (5)$$

Substituting value of a in (5), we have,
$$\tfrac{625}{16}+\tfrac{250}{4}b^2+5b^4=55.$$

Reducing, $$625+1000b^2+80b^4=880.$$

Solving which, we obtain, $b=\pm\frac{1}{2}$, or $\pm\frac{1}{2}\sqrt{-55}$.

Using the values $a=\frac{5}{2}$ and $b=\pm\frac{1}{2}$, we have,
$$a+b=3 \text{ or } 2,$$
$$a-b=2 \text{ or } 3.$$

and
$$\left.\begin{aligned}\therefore\ x&=3 \text{ when } y=2;\\x&=2 \text{ when } y=3.\end{aligned}\right\}$$

Case VI. — Solve the equations:

$$x^4 + x^2y^2 + y^4 = 17\tfrac{1}{16}, \qquad (1)$$

$$x^2 + xy + y^2 = 5\tfrac{1}{4}. \qquad (2)$$

From (1), we have,

$$(x^2 + xy + y^2)(x^2 - xy + y^2) = 17\tfrac{1}{16}. \qquad (3)$$

Dividing (3) by (2), we have,

$$x^2 - xy + y^2 = 3\tfrac{1}{4}. \qquad (4)$$

Subtracting (4) from (2), we have,

$$2xy = 2.$$

Hence, $$xy = 1. \qquad (5)$$

Adding (2) and (5), we have,

$$x^2 + 2xy + y^2 = 6\tfrac{1}{4}.$$

Hence, $$x + y = \pm\tfrac{5}{2}. \qquad (6)$$

Subtracting (5) from (4), we have,

$$x^2 - 2xy + y^2 = 2\tfrac{1}{4}.$$

Hence, $$x - y = \pm\tfrac{3}{2}. \qquad (7)$$

Uniting (6) and (7), we have, $2x = \pm\tfrac{5}{2} \pm \tfrac{3}{2}.$

$$\therefore\ x = \pm 2 \text{ or } \pm\tfrac{1}{2}.$$

$$\therefore\ y = \pm\tfrac{1}{2} \text{ or } \pm 2.$$

EXERCISE 100.

Case I. — Solve the following equations:

1. $\begin{cases} x + 2y = 5, \\ x^2 + y^2 = 5. \end{cases}$

2. $\begin{cases} x^2 - y^2 = 16, \\ xy = 15. \end{cases}$

3. $\begin{cases} x^2 + y^2 = 101, \\ 2xy = 20. \end{cases}$

4. $\begin{cases} x^2 + xy = 168, \\ x + y = 14. \end{cases}$

5. $\begin{cases} 5x + 7y = 74, \\ x^2 + y = 32. \end{cases}$

6. $\begin{cases} 9x^2 = 4y^2, \\ x^2 + 2xy = 21. \end{cases}$

EXERCISE 101.

Case II.

1. $\begin{cases} x+y=19, \\ xy=90. \end{cases}$

2. $\begin{cases} x-y=10, \\ 2xy=150. \end{cases}$

3. $\begin{cases} x^2+y^2=149, \\ x+y=17. \end{cases}$

4. $\begin{cases} x^2+y^2=65, \\ xy=28. \end{cases}$

5. $\begin{cases} x^2-y^2=64, \\ x+y=16. \end{cases}$

6. $\begin{cases} x^2+xy=132, \\ x+y=12. \end{cases}$

7. $\begin{cases} \dfrac{1}{x^2}+\dfrac{1}{y^2}=\dfrac{13}{36}, \\ \dfrac{1}{x}+\dfrac{1}{y}=\dfrac{5}{6}. \end{cases}$

8. $\begin{cases} \dfrac{1}{x^2}-\dfrac{1}{y^2}=\dfrac{5}{36}, \\ \dfrac{1}{x}-\dfrac{1}{y}=\dfrac{1}{6}. \end{cases}$

9. $\begin{cases} x^2=4y^2, \\ x^2+y=105. \end{cases}$

10. $\begin{cases} x+y=25, \\ \sqrt{xy}=12. \end{cases}$

11. $\begin{cases} x-y=64, \\ \sqrt{x}+\sqrt{y}=4. \end{cases}$

12. $\begin{cases} x-\sqrt{y}=30, \\ \sqrt{x}+\sqrt[4]{y}=10. \end{cases}$

13. $\begin{cases} \dfrac{1}{\sqrt{x}}+\dfrac{1}{\sqrt{y}}=\dfrac{11}{30}, \\ \dfrac{1}{x}-\dfrac{1}{y}=\dfrac{11}{900}. \end{cases}$

14. $\begin{cases} x^{\frac{1}{2}}+x^{\frac{1}{4}}y^{\frac{1}{4}}=20, \\ x^{\frac{1}{4}}-x^{\frac{1}{2}}y^{\frac{1}{2}}=2. \end{cases}$

15. $\begin{cases} \dfrac{1}{x^2}+\dfrac{1}{y^2}=13, \\ \dfrac{1}{x}+\dfrac{1}{y}=6. \end{cases}$

EXERCISE 102.

Case III.

1. $\begin{cases} x^2+xy=156, \\ xy+y^2=13. \end{cases}$

2. $\begin{cases} 2x^2+xy+2y^2=32, \\ 3x^2+xy+y^2=27. \end{cases}$

3. $\begin{cases} x^2 + xy + y^2 = 331, \\ x^2 - xy + y^2 = 111. \end{cases}$

4. $\begin{cases} 2x^2 - xy = 28, \\ 2x^2 + 4y^2 = 36. \end{cases}$

5. $\begin{cases} 3x^2 + xy + y^2 = 47, \\ 20x^2 - 15xy - 5y^2 = -195. \end{cases}$

6. $\begin{cases} x^2 + 3xy = 91, \\ xy + 3y^2 = 26. \end{cases}$

7. $\begin{cases} x^2 - 3xy - 10y^2 = 15, \\ x^2 + 5xy + 6y^2 = 255. \end{cases}$

8. $\begin{cases} x^2 + xy + 4y^2 = 6, \\ 3x^2 + 8y^2 = 14. \end{cases}$

9. $\begin{cases} x^2 - 4y^2 = 9, \\ xy + 2y^2 = 3. \end{cases}$

10. $\begin{cases} x^2 + xy = 15, \\ xy - y^2 = 2. \end{cases}$

11. $\begin{cases} x + \sqrt{xy} + y = 10, \\ x - \sqrt{xy} + y = 8. \end{cases}$

12. $\begin{cases} \dfrac{1}{x} + \dfrac{1}{\sqrt{xy}} = \dfrac{1}{2}, \\ \dfrac{1}{\sqrt{xy}} + \dfrac{1}{2y} = \dfrac{3}{8}, \end{cases}$

13. $\begin{cases} 128x^2 + 48xy - 9y^2 = 286, \\ 272x^2 + 72xy - 9y^2 = 814. \end{cases}$

14. $\begin{cases} x^2 - xy + y^2 = 10, \\ 3x^2 + 13xy + 8y^2 = 162. \end{cases}$

15. $\begin{cases} \dfrac{3}{x} + \dfrac{3}{\sqrt{xy}} + \dfrac{5}{y} = 58, \\ \dfrac{11}{x} - \dfrac{5}{\sqrt{xy}} + \dfrac{3}{y} = 26. \end{cases}$

EXERCISE 103.

CASE IV.

1. $\begin{cases} x^4 - y^4 = 65, \\ x - y = 1. \end{cases}$

2. $\begin{cases} x^3 - y^3 = 117, \\ x - y = 2. \end{cases}$

3. $\begin{cases} x^4 + y^4 = 10625, \\ x + y = 15. \end{cases}$

4. $\begin{cases} x^3 + y^3 = 91, \\ x + y = 7. \end{cases}$

5. $\begin{cases} x^4 + x^2y^2 + y^4 = 481, \\ x^4 - 3\,x^2y^2 + y^4 = -95. \end{cases}$

EXERCISE 104.

Case V.

1. $\begin{cases} x^5 + y^5 = 1025, \\ x + y = 5. \end{cases}$

3. $\begin{cases} x^4 + y^4 = 14096, \\ x + y = 18. \end{cases}$

2. $\begin{cases} x^5 - y^5 = 992, \\ x - y = 2. \end{cases}$

4. $\begin{cases} x^4 - y^4 = 9999, \\ x - y = 9. \end{cases}$

5. $\begin{cases} x^2 + y^2 + x + y = 18, \\ 2\,x^2 + y^2 - x + y = 18. \end{cases}$

EXERCISE 105.

Case VI.

1. $\begin{cases} x^4 + x^2y^2 + y^4 = 1281, \\ x^2 + xy + y^2 = 61. \end{cases}$

2. $\begin{cases} x^2 + 3\,xy + 2\,y^2 = 15, \\ x^2 - 2\,xy - 3\,y^2 = -15. \end{cases}$

3. $\begin{cases} x^6 - y^6 = 63, \\ x^4 + x^2y^2 + y^4 = 21. \end{cases}$

4. $\begin{cases} 10\,x^2 - 17\,xy + 3\,y^2 = -42, \\ 15\,x^2 - 13\,xy + 2\,y^2 = 147. \end{cases}$

5. $\begin{cases} 3\,x + \sqrt{xy} - 10\,y = -104, \\ 5\,x + 7\sqrt{xy} - 6\,y = 936. \end{cases}$

MISCELLANEOUS EXAMPLES.

EXERCISE 106.

1. $\left\{\begin{array}{l} x^2 + xy = 15, \\ xy + y^2 = 10. \end{array}\right.$

2. $\left\{\begin{array}{l} x^2 - y^2 = 27, \\ x + y = 27. \end{array}\right.$

3. $\left\{\begin{array}{l} 9y^2 - 12xy = 144 - 16x^2, \\ 3xy = 36. \end{array}\right.$

4. $\left\{\begin{array}{l} x^2 + xy + y^2 = 37, \\ x + y - 7 = 0. \end{array}\right.$

5. $\left\{\begin{array}{l} x^2 - xy + y^2 = 43, \\ x^3 - 2xy(x - y) - y^3 = 43. \end{array}\right.$

6. $\left\{\begin{array}{l} x^2 + xy + y^2 = \frac{3}{4}, \\ x^3 + 2xy(x + y) + y^3 = \frac{3}{4}. \end{array}\right.$

7. $\left\{\begin{array}{l} x^3 + x^2y + xy^2 + y^3 = 13740, \\ x - y - 4 = 0. \end{array}\right.$

8. $\left\{\begin{array}{l} x^3 + y^3 = 1064, \\ x^2 + y^2 = 116. \end{array}\right.$

9. $\left\{\begin{array}{l} x^3 - y^3 = 485, \\ x^2 + y^2 = 73. \end{array}\right.$

10. $\left\{\begin{array}{l} x^3 + y^3 = 1064, \\ xy = 40. \end{array}\right.$

11. $\left\{\begin{array}{l} x^3 - y^3 = 63 \\ xy = 4. \end{array}\right.$

12. $\left\{\begin{array}{l} x^3 + y^3 = 91, \\ x^2 - y^2 = 7. \end{array}\right.$

13. $\left\{\begin{array}{l} 8x^3 + 27y^3 = 2960, \\ 2x + 3y = 20. \end{array}\right.$

14. $\left\{\begin{array}{l} 64x^3 - 8y^3 = 2168, \\ 4x - 2y = 2. \end{array}\right.$

15. $\begin{cases} x^2 - 3xy - 10y^2 = 15, \\ x^2 + 5xy + 6y^2 = 255. \end{cases}$

16. $\begin{cases} x^2 + y^2 - 4(x + y) = 120, \\ xy = 100. \end{cases}$

17. $\begin{cases} x^2 + y^2 + 8(x + y) = 130, \\ 2xy + 4x + 4y = 90. \end{cases}$

18. $\begin{cases} 4x^2y^2 - 8xy = -4, \\ x - y = \frac{3}{2}. \end{cases}$

19. $\begin{cases} (x + y)^2 + (x + y) = 420, \\ x^2 - y^2 = 0. \end{cases}$

20. $\begin{cases} x^4 + 4x^3y + 4y^2 + 8(x^2 + y) = 20945, \\ x^4 + 2x^3y + y^2 = 11980. \end{cases}$

21. $\begin{cases} x^2 + y^2 = 13, \\ x^2 - y^2 = -10.12. \end{cases}$

22. $\begin{cases} 169x^4 + 2y^2 = 177, \\ 4y^2 - 13x^2 = 3. \end{cases}$

23. $\begin{cases} 7x^2 - 11y^2 = 50, \\ x = 3y. \end{cases}$

24. $\begin{cases} (x - 2)(y - 3) = 1, \\ \dfrac{x - 2}{y - 3} = 1. \end{cases}$

25. $\begin{cases} \sqrt{x + y} + \sqrt{x - y} = \sqrt{a}, \\ x^2 - y^2 = b^2. \end{cases}$

26. $\begin{cases} x^3 - y^3 = m(x^2 - xy + y^2), \\ x - y = n. \end{cases}$

27. $\begin{cases} x^2 - y^2 + x + y = .375, \\ x^2 - y^2 - (x - y) = .125. \end{cases}$

28. $\begin{cases} x^2 + y^2 = 25x^2y^2, \\ 24xy = 2. \end{cases}$

29. $\begin{cases} x^2 = ax + by, \\ y^2 = ay + bx. \end{cases}$

30. $\begin{cases} x^3 = ax + by, \\ y^3 = bx + ay. \end{cases}$

31. $\begin{cases} x+y+\sqrt{x+y}=12, \\ x^3+y^3=189. \end{cases}$

33. $\begin{cases} x^4+y^4=m, \\ x+y=c. \end{cases}$

32. $\begin{cases} x^4+y^4=m, \\ x+y=n. \end{cases}$

34. $\begin{cases} x^5-y^5=a, \\ x-y=b. \end{cases}$

EXERCISE 107.

1. Find two numbers whose product is 576, and whose quotient is $\frac{9}{4}$.

2. I have a certain number of silver dollars, which, put in the form of a square, give 29 on each side; if I make two squares, I find that one contains 41 more dollars than the other.

3. Using two given lines as the legs of a right triangle, I obtain an hypothenuse of 17 feet. Using one of these sides as a leg and the other as an hypothenuse, I obtain a side the square of which is 161. How long are the given lines?

4. The area of a certain right triangle is 24 square feet. The area of a square erected on the hypothenuse is 100 square feet. What is the length of each leg?

5. The area of an isosceles trapezoid, in which the upper base equals the altitude, is 42. The area of a triangle, formed on the lower base with one leg and a diagonal, is 24. What is the length of each base?

6. The sum of two numbers is 50; the difference of their squares is 200. What are the numbers?

7. The length of a rectangular field is 119 feet, the width 19. How many feet must be taken from the length, and how many added to the width, in order to increase the perimeter by 24 feet and have the same area?

8. Find three numbers in Arithmetical Progression such that their sum shall be 15 and the sum of their squares 93.

9. The sum of the digits of a two-place number is 7. The digit in the tens' place, plus the square of the digit in the units' place, equals 7. What is the number?

10. The product of the digits of a certain number is equal to one-half of the number itself. If the digits in the units' and tens' places are interchanged, and the given number subtracted from the result, the remainder will be equal to $1\frac{1}{2}$ times the product of the digits. What is the number?

11. The diagonal of a rectangle is 20.4. If I increase the length by 14, and decrease the width by 2.4, the diagonal will be 32.8. What is the length and width of the rectangle?

12. In a distance of 1732.5 yards the front wheel of a cart makes 165 more revolutions than the back wheel. If I increase the circumference of the front wheel by $\frac{3}{4}$ of a yard, its number of revolutions in the same space will be decreased 53. What is the radius of each wheel? What the circumference?

13. A fortress is besieged by the enemy. It is found on calculation that the place can hold out only 12 days, for lack of food. If 120 men leave, and each man is served with $\frac{5}{16}$ of a pound of bread less, the place can hold out 16 days; but if 200 men leave, and each man is served with $\frac{3}{16}$ of a pound less, it can hold out for the same length of time. How many men in the fortress?

CHAPTER XXVIII.

RATIO, PROPORTION, AND VARIATION.

270. Ratio. The relation which one quantity bears to another with respect to magnitude is called their ratio.

Thus, the ratio of 8 to 4 is $\frac{8}{4}$, or 2; and the ratio of a to b is $\frac{a}{b}$, or $a \div b$, or $a : b$.

Here a is called the antecedent, b the consequent, of the ratio.

271. Hence, *every ratio is measured by a fraction, whose numerator is the antecedent of the ratio, and whose denominator is the consequent of the ratio.*

The ratio a to b equals the ratio c to d when $\frac{a}{b} = \frac{c}{d}$.

272. When the terms of a ratio are interchanged, the resulting ratio is called the **Inverse** of the given ratio; thus, $\frac{b}{a}$ is the inverse of $\frac{a}{b}$; or $b : a$ is the inverse of a : b.

273. *If the terms of a ratio are multiplied or divided by the same quantity, the ratio is not altered.*

For, multiplying both terms of $\frac{a}{b}$ by m, we have $\frac{a}{b} = \frac{ma}{mb}$. Hence, $a : b = ma : mb$.

And, dividing both terms of $\frac{a}{b}$ by m, we have $\frac{\frac{a}{m}}{\frac{b}{m}}=\frac{a}{b}$, which reduced gives $\frac{ma}{mb}=\frac{a}{b}$.

274. If the antecedent is greater than the consequent, the ratio is called a **Ratio of Greater Inequality.**

If the antecedent is less than the consequent, the ratio is called a **Ratio of Less Inequality.**

275. If the antecedent is equal to the consequent, the ratio is called a **Ratio of Equality.**

276. Ratios are **compared** by comparing the fractions.

Thus, $a:b >$ or $< c:d$,

as $\frac{a}{b} >$ or $< \frac{c}{d}$;

as $\frac{ma}{mb} >$ or $< \frac{m'c}{m'd}$,

as $ma >$ or $< m'c$.

277. Ratios are **compounded** by taking the product of the fractions which represent them.

Thus, the ratio compounded of $a:b$, $c:d$, and $e:f$, is $ace:bdf$.

$a^2:b^2$ is the duplicate ratio of $a:b$.

$a^3:b^3$ is the triplicate ratio of $a:b$.

278. Proportion. *A proportion is an equality of ratios.*

Thus, $a:b=c:d$ is a proportion.

279. The four letters, a, b, c, and d, in the proportion, $a:b=c:d$, are called proportionals.

a and d are called the **Extremes.**

b and c are called the **Means.**

280. In a proportion in which the means are equal, either mean is called a **Mean Proportional** between the first and last terms.

Thus, in $a:b=b:c$, b is a mean proportional between a and c.

281. In a proportion in which the means are equal, the last term is called a **Third Proportional** to the first and second terms.

Thus, in $a:b:b:c$, c is a third proportional to a and b.

282. A **Fourth Proportional** to three quantities is the fourth term of a proportion, whose first three terms are the three quantities taken in their order.

283. A **Continued Proportion** is a series of equal ratios in which each consequent is the same as the following antecedent.

PROPERTIES OF PROPORTIONS.

284. I. *When four quantities are in proportion, the product of the means equals the product of the extremes.*

Let a, b, c, d, be in proportion.

Then $$\frac{a}{b}=\frac{c}{d}.$$

Clearing of fractions, $$ad=cb.$$

From this, we have, $$a=\frac{cb}{d};\quad d=\frac{cb}{a};$$

$$c=\frac{ad}{b};\quad b=\frac{ad}{c}.$$

That is, either extreme equals the product of the means by the other extreme, and vice versa.

285. II. *If the product of two quantities is equal to the product of two other quantities, either set may be made the extremes of a proportion, and the other set the means.*

Let $ad = cb$.

Dividing by bd,
$$\frac{ad}{bd} = \frac{cb}{bd},$$

or,
$$\frac{a}{b} = \frac{c}{d}.$$

$$\therefore\ a : b = c : d.$$

286. III. **A mean proportional** *between two quantities is equal to the square root of their product.*

Let the proportion be $a : b = b : c$.

Then
$$\frac{a}{b} = \frac{b}{c},$$

and
$$b^2 = ac.$$

Taking the square root of each member of the last equation, we have,
$$b = \sqrt{ac}.$$

287. IV. **Alternation.** *If four quantities are in proportion, they are in proportion by alternation.*

Let the proportion be $a : b = c : d$.

Then
$$\frac{a}{b} = \frac{c}{d}.$$

Multiplying both members by $\frac{b}{c}$, we have,
$$\frac{ab}{bc} = \frac{bc}{cd},$$

or,
$$\frac{a}{c} = \frac{b}{d}.$$

$$\therefore\ a : c = b : d.$$

That is, the first term is to the third as the second is to the fourth.

288. V. **Inversion.** *If four quantities are in proportion, they are in proportion by inversion.*

Let $a:b=c:d.$

Then $\frac{a}{b}=\frac{c}{d}.$

Taking the reciprocals of both members, we have,

$$\frac{b}{a}=\frac{d}{c}.$$

$$\therefore\ b:a=d:c.$$

That is, the second term is to the first as the fourth is to the third.

289. VI. **Composition.** *If four quantities are in proportion, they are in proportion by composition.*

If $a:b=c:d.$

To prove $a+b:b=c+d:d.$

Let $a:b=c:d.$

Then, $\frac{a}{b}=\frac{c}{d}.$

Adding 1 to each member, we have,

$$\frac{a}{b}+1=\frac{c}{d}+1,$$

and $\frac{a+b}{b}=\frac{c+d}{d}.$

$$\therefore\ a+b:b=c+d:d.$$

290. VII. **Division.** *If four quantities are in proportion, they are in proportion by division.*

If $a:b=c:d.$

To prove $a-b:b=c-d:d.$

Let $a:b=c:d.$

Then, $\frac{a}{b}=\frac{c}{d}.$

Subtracting 1 from each member, we have,

$$\frac{a}{b}-1=\frac{c}{d}-1,$$

and $\frac{a-b}{b}=\frac{c-d}{d}.$

$$\therefore\ a-b:b=c-d:d.$$

291. VIII. Composition and Division. *If four quantities are in proportion, they are in proportion by composition and division.*

If $a:b=c:d.$

To prove $a+b:a-b=c+d:c-d.$

By composition, $\frac{a+b}{b}=\frac{c+d}{d}.$ (1)

By division, $\frac{a-b}{b}=\frac{c-d}{d}.$ (2)

Dividing (1) by (2), we have,

$$\frac{a+b}{a-b}=\frac{c+d}{c-d}.$$

$$\therefore\ a+b:a-b=c+d:c-d.$$

292. IX. In a Series of Equal Ratios *the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.*

If $a:b=c:d=e:f=g:h.$

To prove $(a+b+e+g):(b+d+f+h)=a:b.$

If $a:b=c:d=e:f=g:h,$

then, $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{g}{h}.$

Let r represent the ratio.

Then, $$\frac{a}{b}=r,\ \frac{c}{d}=r,\ \frac{e}{f}=r,\ \frac{g}{h}=r,$$

and $$a=br,\ c=dr,\ e=fr,\ g=hr.$$

By addition, $$a+c+e+g=br+dr+fr+hr,$$

and $$\frac{a+c+e+g}{b+d+f+h}=r=\frac{a}{b}.$$

$$\therefore\ a+c+e+g:b+d+f+h=a:b.$$

293. X. *The products of the corresponding terms of two or more proportions are in proportion.*

For if $$\begin{cases} a:b=c:d, \\ e:f=g:h, \\ i:k=l:m, \end{cases}$$

then $$\begin{cases} \dfrac{a}{b}=\dfrac{c}{d}, \\ \dfrac{e}{f}=\dfrac{g}{h}, \\ \dfrac{i}{k}=\dfrac{l}{m}. \end{cases}$$

By multiplication, $$\frac{aei}{bfk}=\frac{cgl}{dhm}.$$

$$\therefore\ aei:bfk=cgl:dhm.$$

294. XI. *If four quantities are in proportion, like powers and like roots of these quantities are in proportion.*

Let $$a:b=c:d.$$

Then $$\frac{a}{b}=\frac{c}{d},$$

and $$\frac{a^n}{b^n}=\frac{c^n}{b^n}, \tag{1}$$

also $$\frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}=\frac{c^{\frac{1}{n}}}{b^{\frac{1}{n}}}. \tag{2}$$

$\therefore$ from (1), $a^n : b^n = c^n : d^n$.

$\therefore$ from (2), $\sqrt[n]{a} : \sqrt[n]{b} = \sqrt[n]{c} : \sqrt[n]{d}$.

295. XII. *If four quantities are in proportion, either the antecedents or the consequents may be multiplied or divided by any quantity and the result will be in proportion.*

Let $a : b = c : d$.

Then $\frac{a}{b} = \frac{c}{d}$,

and $$\frac{na}{b} = \frac{nc}{d}. \qquad (1)$$

Also, $$\frac{a}{nb} = \frac{c}{nd}. \qquad (2)$$

Now, if n is integral, we have,

from (1), $na : b = nc : d$; (3)

from (2), $a : nb = c : nd$. (4)

296. Putting $n = \frac{1}{m}$, we have, from (3) and (4),

$$\frac{a}{m} : b = \frac{c}{m} : d, \qquad (5)$$

and $$a : \frac{b}{m} = c : \frac{d}{m}. \qquad (6)$$

297. XIII. *If four quantities are in proportion, equimultiples of one couplet are in proportion to equimultiples of the other couplet.*

Let $a : b = c : d$.

Then $\frac{a}{b} = \frac{c}{d}$,

and $$\frac{ma}{mb} = \frac{nc}{nd}.$$

$$\therefore ma : mb = nc : nd.$$

298. XIV. *If four quantities in proportion be multiplied or divided, term by term, by four other quantities also in proportion, the results, taken in order, will be in proportion.*

Let $a : b = c : d,$ (1)

and $e : f = g : h.$ (2)

Then from (1), $ad = bc,$ (3)

and from (2), $eh = fg.$ (4)

Multiplying (3) by (4), we have,

$$(ad)(eh) = (bc)(fg),$$

$$(ae)(dh) = (bf)(cg).$$

$$\therefore\ ae : bf = cg : dh.$$

Dividing (3) by (4),

$$\frac{ad}{eh} = \frac{bc}{fg};$$

or,

$$\left(\frac{a}{e}\right)\left(\frac{d}{h}\right) = \left(\frac{b}{f}\right)\left(\frac{c}{g}\right).$$

$$\therefore\ \frac{a}{e} : \frac{b}{f} = \frac{c}{g} : \frac{d}{h}.$$

299. XV. *If three quantities are in continued proportion, the first is to the third as the square of the first is to the square of the second.*

Let $a : b = b : c.$

Then, $b^2 = ac.$

Multiplying by a, $ab^2 = a^2c.$

$$\therefore\ a : c = a^2 : b^2.$$

300. XVI. *If four quantities are in continued proportion, the first is to the fourth as the cube of the first is to the cube of the second.*

Let $$a : b = b : c = c : d.$$

Then, $$\frac{a}{b} = \frac{b}{c} = \frac{c}{d}.$$

$$\therefore\ \frac{a}{b} \times \frac{b}{c} \times \frac{c}{d} = \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b}.$$

$$\therefore\ \frac{a}{d} = \frac{a^3}{b^3}.$$

$$\therefore\ a : b = a^3 : b^3.$$

Examples:

(1) If $$a : b = c : d,$$

show that $$\frac{a^2 + 2b + bm}{2b + bm} = \frac{ac + 2d + dm}{2d + dm}.$$

Separating the members, we have,

$$\frac{a^2}{2b + bm} + \frac{2b + bm}{2b + bm} = \frac{ac}{2d + dm} + \frac{2d + dm}{2d + dm},$$

and $$\frac{a^2}{2b + bm} + 1 = \frac{ac}{2d + dm} + 1.$$

Subtracting 1 from each member, and multiplying by $\frac{2 + m}{a}$,

we have, $$\frac{a}{b} = \frac{c}{d};$$

or, $$a : b = c : d.$$

(2) Find x when

$$\frac{x + m}{2x - n} = \frac{3x + n}{4x - m}.$$

Clearing of fractions, we have,

$$4x^2 + 3mx - m^2 = 6x^2 - nx - n^2,$$

and $$2x^2 - x(3m + n) = n^2 - m^2,$$

and $$16x^2 - 8x(3m + n) = 8n^2 - 8m^2.$$

Completing the square and uniting,

$$16x^2 - 8x(3m + n) + (3m + n)^2 = 9n^2 + 6mn + m^2.$$

Taking the square root,

$$4x - (3m + n) = \pm(m + 3n).$$

$$\therefore\ x = m + n, \text{ or } \frac{m - n}{2}.$$

EXERCISE 108.

1. If $a : b = c : d$, show that

$$a^2 - b^2 : b^2 = c^2 - d^2 : d^2.$$

2. If $a : b = c : d$, show that

$$3a^2 + ab + 2b^2 : 3a^2 - 2b^2 = 3c^2 + cd + 2d^2 : 3c^2 - 2d^2.$$

3. If $a : b = c : d$, show that

$$a^2 + b^2 : c^2 + d^2 = a^2 - b^2 : c^2 - d^2.$$

4. If $a : b = c : d$, show that

$$\frac{a^3 + ac^2 + a^2c + c^3}{a^2 - ac^2 - a^2c + c^3} = \frac{b^3 + bd^2 + b^2d + d^3}{b^3 - bd^2 - b^2d + d^3}.$$

5. The last three terms of a proportion are 5, 12, and 15. What is the first term?

6. Find a third proportional to 4 and 25.

7. Find a fourth proportional to 6, 10, 27.

8. Find a third proportional to $\frac{a}{b} + \frac{b}{a}$, and $\frac{a}{b}$.

9. If $a : b = c : d$, show that

$$a - c : b - d = \sqrt{a^2 + c^2} : \sqrt{b^2 + d^2}.$$

10. If $a : b = c : d = e : f$, and r equals the ratio, show that

$$\sqrt{\frac{a^2 + b^2 + e^2}{b^2 + d^2 + f^2}} = r.$$

11. If $a : b = m : n$, show that,

$$a^2 + b^2 : \frac{a^3}{a+b} = m^2 + n^2 : \frac{m^3}{m+n}.$$

Solve the equations:

12. $\dfrac{3x^4 + x^3 - 2x - 3}{3x^4 - x^3 + 2x + 3} = \dfrac{5x^4 + 2x^3 - 7x + 3}{5x^4 - 2x^3 + 7x - 3}.$

13. $\dfrac{x + \sqrt{1 - x^2}}{x - \sqrt{1 - x^2}} = \dfrac{a + \sqrt{b^2 - a^2}}{a - \sqrt{b^2 - a^2}}.$

14. $\begin{cases} \dfrac{x-y}{x+y} = \dfrac{2}{9}, \\ \dfrac{x+y}{xy} = \dfrac{18}{77}. \end{cases}$

VARIATION.

301. When two quantities are so related that the ratio of any two values of the first equals the ratio of the corresponding values of the second, they are said to **vary directly.**

302. A quantity which has a fixed value is called a **Constant Quantity,** or a **Constant.**

303. A quantity which may change its value is called a **Variable Quantity,** or a **Variable.**

The letters, x, y, z, etc., generally represent variable numbers.
The letters a, b, c, etc., generally represent constant numbers.

304. One quantity is said to **vary inversely** as another when the first varies directly as the reciprocal of the second.

305. One quantity is said to vary as two others **jointly** when it varies directly as the product of the two.

306. One quantity is said to vary **directly** as a second, and **inversely** as a third, when it varies jointly as the second and the reciprocal of the third.

Thus, by attraction of gravitation, a body varies directly as the mass, and inversely as the square of the distance.

The symbol of variation is $\propto$. Thus, $x \propto y$ is read "x varies as y."

307. I. *If $x \propto y$, then y multiplied by some constant quantity equals x.*

Let x' and y' denote fixed corresponding values of x and y, and x and y any other pair.

Then, $$\frac{x}{x'} = \frac{y}{y'}, \text{ or } x = \frac{x'}{y'}y.$$

Letting $\frac{x'}{y'} =$ the constant ratio m, we have, $x = my$.

It, therefore, follows:

308. II. *If x varies inversely as y, that $x = \frac{m}{y}$.*

309. III. *If x varies jointly as y and z, that $x = myz$.*

310. IV. *If x varies directly as y and inversely as z, that $x = \frac{my}{z}$.*

THEOREMS:

311. I. **If $y \propto x$ and $x \propto z$, then $y \propto z$.**

For $$y = mx,$$
$$x = nz.$$
$$\therefore\ xy = mnxz,$$
and $$y = mnz.$$
$$\therefore\ y \propto z.$$

312. II. **If $y \propto x$ and $x \propto z$, then $(y \pm z) \propto x$.**

For $$y = mx,$$
$$z = nx.$$

$$\therefore\ y \pm z = mx \pm nx.$$

$$\therefore\ y \pm z = (m \pm n)x.$$

$$\therefore\ y \pm z \propto x.$$

313. III. **If z is constant when $x \propto y$, and x is constant when $y \propto z$, $y \propto xz$ when x and z are variables.**

Let x', y', z', and x'', y'', z'' be fixed corresponding sets of variables.

Let x become x' and x'', while z is constant, and let the corresponding value of y be Y.

Then $$y' : Y = x' : x''. \qquad (1)$$

Now let z become z' and z'', while x remains constant.

Then $$Y : y'' = z' : z''. \qquad (2)$$

Then, from (1) and (2), we have,

$$y'Y : y''Y = x'z' : x''z'';$$

also, $$y' : y'' = x'z' : x''z'';$$

also, $$y' : x'z' = y'' : x''z''.$$

$\therefore$ the ratio $\frac{Y}{xz}$ is constant, and $y \propto xz$.

Thus, the area of a rectangle varies as the altitude when the base is constant, as the base when the altitude is constant, and as the product of the base and altitude when both are variable.

Accordingly, a rectangular solid will vary as its length, width, and thickness.

EXAMPLES:

(1) If x varies jointly as y and z, and $x = 3$, $y = 4$, $z = 5$, find x when $y = 10$, $z = 10$.

We have, $$x = myz,$$

and $$m = \frac{x}{yz}.$$

Substituting values, we have,

$$m = \frac{3}{20}.$$

$$\therefore\ x = 15.$$

(2) In 2 seconds a body falls 64* feet. How far will a body fall in 3 seconds, if the distance varies as the square of the time?

$T =$ time; $d =$ distance.

$$\therefore\ d \propto T^2.$$

Let $d = mT^2$,

then $64 = mT^2$,

and $m = 16.$

$$\therefore\ d = 16 \times 3^2 = 144.$$

EXERCISE 109.

Solve:

1. $M \propto n$ and $p \propto q$, then will $mp \propto nq$.

2. If $x \propto \frac{y}{z}$ when $x = 6$, $y = 4$, and $z = 3$, find the value of x when $y = 5$ and $z = 7$.

3. If z varies jointly as x and y, and $z = 4$ when $x = 1$ and $y = 2$, find the value of x when $z = 30$ and $y = 3$.

4. If $x^2 \propto y^3$, and $x = 3$ when $y = 4$, find the equation between x and y.

5. Show that the altitudes vary inversely as the bases in triangles of equal area.

6. Show that in parallelograms of equal area the altitudes vary inversely as the bases.

7. If $x \propto y$, show that $x^2 + y^2 \propto x^2 - y^2$.

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* Approximately.

8. The volume of a pyramid whose base is 9 square feet and whose altitude is 10 feet contains 270 cubic feet; what must be the altitude of a pyramid of a base of 3 square feet, that it may contain 54 cubic feet?

Note.— The volume varies as the base and altitude jointly.

9. The volume of a sphere varies as the cube of its radius. When the radius is 1 foot and 6 inches the volume is 14$\frac{1}{7}$ cubic feet; find the volume of a sphere when the radius is 2 feet and 6 inches.

10. The space through which a body falls varies as the square of the time occupied in falling; if a body falls 400 feet in 5 seconds, how far will it fall in 8 seconds?

11. The area of a circle varies as the square of the radius. If the area of a circle with a radius of 3$\frac{1}{2}$ feet is 38$\frac{1}{2}$ square feet, what is the area of a circle whose radius is 4$\frac{1}{2}$ feet?

CHAPTER XXIX.

ARITHMETICAL PROGRESSION.

SERIES.

314. When numbers proceed in succession according to a fixed law they form a **Series.**

Each number is called a term of the series.

315. A **Finite Series** ends at some particular term.

316. An **Infinite Series** continues without end.

317. An **Arithmetical Series** is formed by adding to each preceding term a constant number called the **Common Difference.**

318. In an arithmetical series,

$a =$ the first term.
$d =$ the common difference.
$l =$ the last term.
$n =$ the number of terms.
$S =$ the sum of all the terms.

Thus, $a, a + d, a + 2d, a + 3d, \cdots$ is the general representation of an arithmetical series.

319. I. **To find the last term, *l*, when the first term, *a*, the common difference, *d*, and the number of terms, *n*, are given.**

320. In the progression, $a, a + d, a + 2d, a + 3d, \cdots$ the coefficient of d in any term is 1 less than the number of terms. Hence, the nth or last term will be $a + (n - 1)d$.

That is, $$l = a + (n - 1)d. \qquad \text{(I.)}$$

321. II. **To find the sum, *S*, when the first term, *a*, the last term, *l*, and the number of terms, *n*, are given.**

$$S = a + (a + d) + (a + 2d) + \cdots + (l - d) + l,$$

or $$S = l + (l - d) + (l - 2d) + \cdots + (a + d) + a.$$

Adding these equations, we have,

$$2S = (a + l) + (a + l) + (a + l) + \cdots + (a + l) + (a + l),$$

or $$2S = n(a + l).$$

Dividing by 2, $$S = \frac{n}{2}(a + l) \qquad \text{(II.)}$$

Substituting in (II.) the value of l in formula (I.), we have,

$$S = \frac{n}{2}(a + a + (n - 1)d),$$

$$= \frac{n}{2}(2a + (n - 1)d). \qquad \text{(III.)}$$

322. The **Arithmetical Mean** between two numbers is the number which stands between them, and which makes an Arithmetical Series with them.

If A represents the arithmetical mean, we shall have a, A, b representing an arithmetical series.

Hence, $$A - a = b - A,$$

and $$2A = a + b,$$

or $$A = \frac{a + b}{2}. \qquad \text{(IV.)}$$

323. Therefore, *the arithmetical mean between two numbers is equal to one-half their sum.*

Examples:

(1) Given $a = 3$, $n = 10$, $d = -\frac{1}{2}$, to find l.

$$l = a + (n - 1)d.$$

Hence, $$l = 3 + 9 \times (-\tfrac{1}{2})$$

$$= -1\tfrac{1}{2}.$$

(2) Given $a = \frac{1}{2}$, $l = 50$, $n = 15$, to find d.

$$l = a + (n - 1)d.$$

$$\therefore\ d = \frac{l - a}{n - 1}.$$

Hence,
$$d = \frac{49\frac{1}{2}}{14}$$
$$= \tfrac{99}{28}.$$

(3) Given $a = 1$, $n = 10$, $l = 20$, to find the sum, S.

$$S = \frac{n}{2}(a + l).$$

Hence,
$$S = 5 \times 21$$
$$= 105.$$

(4) Given $a = \frac{1}{2}$, $d = 2$, $l = 10$, to find n and S.

From formula (I.) we have,

$$n = \frac{l - a + d}{d}.$$

Substituting this value of n in formula (II.), we have,

$$S = \frac{l - a + d}{2d}(a + l).$$
$$= \tfrac{23}{4} \times \tfrac{21}{2}$$
$$= \tfrac{483}{8}.$$

(5) Given d, l, and S, to find a.

From formula (I.) we have,

$$n = \frac{l - a + d}{d}.$$

From formula (II.) we have,

$$n = \frac{2S}{a + l}.$$

Hence, $$\frac{l-a+d}{d} = \frac{2S}{a+l},$$

and $$l^2 - a^2 + ad + ld = 2dS,$$

$$a^2 - ad = l^2 + ld - 2dS,$$

$$a^2 - (\quad) + (\tfrac{1}{2}d)^2 = l^2 + ld - 2dS + (\tfrac{1}{2}d)^2,$$

$$a - \tfrac{1}{2}d = \pm\sqrt{l^2 + ld - 2dS + (\tfrac{1}{2}d)^2}.$$

$$\therefore\ a = \tfrac{1}{2}d \pm \sqrt{(l + \tfrac{1}{2}d)^2 - 2dS}.$$

(6) Find A when $a = \frac{1}{3}$ and $l = -\frac{1}{10}$.

Formula (IV.) gives $\dfrac{a+l}{2} = A$.

Here we substitute l for b of formula.

Hence, $$\frac{\frac{1}{3} + (-\frac{1}{10})}{2} = A = \tfrac{7}{60}.$$

(7) Find d, when n, l, and S are given.

$$l = a + (n-1)d. \quad \therefore\ a = l - (n-1)d.$$

$$S = \tfrac{1}{2}n(a+l). \qquad \therefore\ a = \frac{2S - ln}{n}.$$

$$\therefore\ l - (n-1)d = \frac{2S - ln}{n},$$

$$ln - dn^2 + dn = 2S - ln$$

$$-dn^2 + dn = 2S - 2ln, \text{ and } d(d-1)n = 2(ln - S).$$

$$\therefore\ d = \frac{2(ln - S)}{n(n-1)}.$$

324. When any three of the letters a, d, l, n, and S are given, the other two may be found by means of the equations:

$$a = a + (n-1)\,d,$$

and $$S = \frac{n}{2}\left(a + l\right).$$

The table on page 232 will be found useful for reference: The student should prove each of these formulas for himself.

No.	Given.	Find.	Answers.
1	$d\,n\,l$	a	$a = l - (n-1)d.$
2	$d\,n\,s$		$a = \frac{s}{n} - \frac{(n-1)d}{2}.$
3	$d\,l\,s$		$a = \frac{1}{2}d \pm \sqrt{(l + \frac{1}{2}d)^2 - 2ds}.$
4	$n\,l\,s$		$a = \frac{2s}{n} - l.$
5	$a\,l\,s$	d	$d = \frac{l^2 - a^2}{2s - l - a}.$
6	$a\,n\,s$		$d = \frac{2(s - an)}{n(n-1)}.$
7	$a\,n\,l$		$d = \frac{l - a}{n - 1}.$
8	$n\,l\,s$		$d = \frac{2(nl - s)}{n(n-1)}.$
9	$a\,d\,n$	l	$l = a + (n-1)d.$
10	$a\,d\,s$		$l = -\frac{1}{2}d \pm \sqrt{2ds + (a - \frac{1}{2}a)^2}.$
11	$a\,n\,s$		$l = \frac{2s}{n} - a.$
12	$d\,n\,s$		$l = \frac{s}{n} + \frac{(n-1)d}{2}.$
13	$a\,d\,l$	n	$n = \frac{l - a}{d} + 1.$
14	$a\,d\,s$		$n = \frac{d - 2a \pm \sqrt{8ds + (2a - d)^2}}{2d}.$
15	$a\,l\,s$		$n = \frac{2s}{l + a}.$
16	$d\,l\,s$		$n = \frac{2l + d \pm \sqrt{(2l + d)^2 - 8ds}}{2d}.$
17	$a\,d\,n$	s	$s = \frac{1}{2}n(2a + dn - d).$
18	$a\,d\,l$		$s = \frac{1}{2}\left[(l + a) + \left(\frac{l^2 - a^2}{d}\right)\right].$
19	$a\,n\,l$		$s = \frac{n}{2}(l + a).$
20	$d\,n\,l$		$s = \frac{1}{2}n(2l - dn + d).$

EXERCISE 110.

1. Find the 9th term of 1, 5, 9 ⋯.

2. Find the 51st term of $-\frac{1}{2}, 1, 2\frac{1}{2} \cdots$.

3. Find the 23d term of $-4, -1, +2 \cdots$.

4. Find the 10th term of $-\frac{1}{10}, \frac{1}{5}, \frac{1}{2} \cdots$.

5. Find the 8th term of $a+2b, a+4b \cdots$.

6. Find the 99th term of $-48, -43, -38 \cdots$.

7. What is the sum of the first 1000 numbers?

8. $a=17,\ n=79,\ d=5\frac{1}{2}$. Find l.

9. $l=24,\ d=\frac{5}{7},\ n=22$. Find a.

10. $a=-6,\ d=\frac{3}{4},\ s=146\frac{1}{4}$. Find n.

11. Find n and s when $a=-\frac{3}{4}$, $p=-\frac{7}{8}$, and $l=21\frac{3}{4}$.

12. $d=.27,\ l=18.53,\ s=628.43$.

13. Find the sum of:

$8+9+10+11+$ to 50 terms,
$-4+(-1)+2+$ to 20 terms,
$\frac{2}{3}+2+3\frac{1}{3}+$ to 10 terms,
$x+(x+y)+(x+2y)+$ to 11 terms,
$a+(a+3b)+(a+6b)+$ to 5 terms.

14. The 7th term is -6, the 37th term is $15\frac{3}{4}$. Find the first term.

15. Place 8 means between 7 and 13.

16. The sum of the 19th, 43d, and 57th terms is 827. The sum of the 27th, 58th, and 69th is 1581. What is the first term.

17. Two arithmetical progressions have the same first term. In one of these $l=20$, $s=50$. In the other $l=35$, $s=80$. Find n.

18. Two arithmetical progressions have the same last term. In one $a = 5$, $s = 75$. In the other $a = 1$, $s = 88$. Find n in each progression.

19. Find the middle term in a progression of 7 terms whose sum is 112.

20. How far does a body fall in the 10th second, if it falls 16 feet the first second, 48 feet the second second, and 80 feet the third second.

21. The arithmetical mean between two numbers is 6, and the mean between 3 times the first and 9 times the last is 45. Find the number.

22. Insert 10 arithmetical means between $-2\frac{1}{4}$ and $\frac{1}{2}$.

23. The sum of three numbers in arithmetical progression is 21, and the sum of their squares is 155. Find the numbers.

24. The 5th term of an arithmetical progression is $-\frac{3}{4}$; the 10th term is $\frac{1}{2}$. What is the first term?

25. When $d = 3$, and the product of the first and third terms equals 10.2 times the second term, what are the first 5 terms?

26. A number consists of three digits which are in arithmetical progression. This number divided by the sum of the digits equals 14.5. If 9 be added to the number, the digits in the tens' and units' places are interchanged.

27. A man hires a servant with the agreement that he gets $ 105 the first year, and each following year $ 5 more than the previous year. How much does the servant get the eleventh year, and how much does he get in all?

28. A contract is made to construct a shaft to a coal mine. It is as follows: $ 3.24 for the first yard, and for each following yard five cents extra. If the shaft is 500 yards deep, how much is paid for the last yard, and how much paid for the construction of the whole?

CHAPTER XXX.

GEOMETRICAL PROGRESSION.

325. A series obtained by multiplying each preceding term by a fixed constant is called a **Geometrical Series** or a **Geometrical Progression.**

326. In the geometrical series

$a =$ the first term,
$l =$ the last term,
$n =$ the number of terms,
$r =$ the ratio,
$s =$ the sum of all the terms.

327. $a + ar + ar^2 + ar^3$, etc., represents a geometrical series.

It will readily be seen that the terms decrease or increase in value as the r is greater or less than unity; also, that if r is negative, the signs of the terms will be alternately positive and negative, or *vice versa.*

328. To find the last term, l, when the first term, a, the ratio, r, and the number of terms, n, are given.

329. In a, ar, ar^2, ar^3 we see that the exponent of r is always one less than the number of the term, hence the exponent of the last term will be $n - 1$.

$$\therefore\ l = ar^{n-1}. \qquad \text{(I.)}$$

330. To find the sum, s, when the first term, a, the last term, l, and the ratio, r, are given.

331. $s = a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1}$. (1)

Multiplying both members by r,

$$rs = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n. \qquad (2)$$

Subtracting (1) from (2),

$$rs - s = ar^n - a,$$

or,

$$s(r-1) = ar^n - a.$$

$$\therefore\ s = \frac{ar^n - a}{r-1}. \qquad \text{(II.)}$$

332. From formula (I.), multiplying both members by r, we have,

$$rl = ar^n.$$

Substituting this value of ar^n in formula (II.), we have,

$$s = \frac{rl - a}{r-1}. \qquad \text{(III.)}$$

333. Geometrical Mean. A geometrical mean between two numbers is a number which makes a geometrical series with the given numbers when placed between them.

334. If G represents the geometrical mean, and a and b the two numbers, then a, G, b make a geometrical series, and we have,

$$\frac{G}{a} = \frac{b}{G}.$$

$$\therefore\ G = \sqrt{ab}. \qquad \text{(IV.)}$$

INFINITE GEOMETRICAL SERIES.

335. In a decreasing progression — that is, when r is less than one — the successive terms become smaller and smaller; therefore, if we take an infinitely large number of terms, the value of the last term will be infinitely small, or less than any assignable quantity.

Explanation of the symbols 0, ∞, and $\frac{0}{0}$.

336. The symbol 0 is called **Zero,** and means either absolute zero or a quantity less than any assignable quantity.

If a be subtracted from a, zero absolute will result.

Thus, $a - a = 0 =$ absolute zero.

337. The ∞ is called **Infinity,** and is used to denote a quantity larger than any assignable quantity.

Now, if $\frac{a}{b}$ be a fraction, and the numerator remains constant while the denominator constantly decreases in value, the value of the fraction increases. When the denominator becomes less than any assignable quantity, the value of the fraction becomes larger than any assignable quantity; that is, when the denominator becomes 0, the value of the fraction becomes ∞, and we have,

$$\frac{a}{0} = \infty. \qquad (1)$$

If the numerator remains constant while the value of the denominator increases to a quantity larger than any assignable quantity, the value of the fraction becomes less than any assignable quantity, and we have,

$$\frac{a}{\infty} = 0. \qquad (2)$$

338. From either of the above equations, (1) and (2), it becomes evident that 0, multiplied by infinity, gives a finite quantity.

The zero in these equations must be taken not as zero absolute, but an infinitely small quantity.

339. The symbol $\frac{0}{0}$ results from $a = 0$ and $b = 0$ in the fraction $\frac{a}{b}$ and is indeterminate, since the denominator, multiplied by any finite quantity whatsoever, gives the result 0.

Hence it is called the **Symbol of Indetermination.**

340. In the equation $s = \frac{rl - a}{r - 1} = \frac{a - rl}{1 - r}$, when $r < 1$, rl will approach the limit zero if the number of terms increases indefinitely, and $\frac{a - rl}{1 - r}$ may be written $\frac{a}{1 - r}$.

Therefore, the sum of a decreasing Geometrical Progression to infinity may be obtained from the formula:

$$s = \frac{a}{1 - r}. \tag{V.}$$

Examples.

(1) Given $a = 2$, $r = \frac{1}{4}$, $n = 7$, to find l.

$$\begin{aligned} l &= ar^{n-1}. \\ \therefore\ l &= 2 \times (\tfrac{1}{4})^6 \\ &= \frac{2}{4^6} \\ &= \tfrac{2}{4096} \\ &= \tfrac{1}{2048}. \end{aligned}$$

(2) Given $a = -\frac{1}{2}$, $r = \frac{1}{3}$, $n = 6$, to find l.

$$\begin{aligned} l &= ar^{n-1}. \\ \therefore\ l &= -\tfrac{1}{2} \times (\tfrac{1}{3})^5 \\ &= -\tfrac{1}{486}. \end{aligned}$$

(3) Given $a = 2$, $r = -\frac{1}{2}$, $l = -\frac{1}{4}$, to find s.

$$s = \frac{rl - a}{r - 1}.$$

$$\therefore\ s = \frac{\frac{1}{8} - 2}{-\frac{1}{2} - 1}$$

$$= \tfrac{5}{4}.$$

(4) Find the sum of a series of five terms whose last term is $\frac{1}{128}$ and whose third term is $\frac{1}{8}$.

Now, for the purpose of finding the ratio, consider the last three terms.

Then, $$l = ar^{n-1},$$

and $$r^{n-1} = \frac{l}{a},$$

$$r^2 = \frac{\frac{1}{128}}{\frac{1}{8}}$$

$$= \tfrac{1}{16}.$$

$$\therefore\ r = \tfrac{1}{4}.$$

Now, from the equations,

$$l = ar^{n-1},$$

and $$s = \frac{a(r^n - 1)}{r - 1},$$

we obtain, $$s = \frac{lr^n - l}{r^n - r^{n-1}}.$$

Substituting the given values,

$$s = \frac{\frac{1}{128}\left(\frac{1}{4}\right)^5 - \frac{1}{128}}{\left(\frac{1}{4}\right)^5 - \left(\frac{1}{4}\right)^4}$$

$$= 2\tfrac{85}{128}.$$

341. The above may be considered the general solution examples of this class. A simple method is as follows:

Consider the last three terms, and

$$G = \sqrt{ab}.$$

$$\therefore\ G = \sqrt{\tfrac{1}{8} \times \tfrac{1}{128}} = \tfrac{1}{32}.$$

$$\therefore\ r = \tfrac{1}{4}.$$

Then considering all the terms,

$$l = ar^{n-1}.$$

Whence $$a = \frac{l}{r^{n-1}} = \frac{\tfrac{1}{128}}{\tfrac{1}{256}}.$$

$$\therefore\ a = 2.$$

$\therefore$ the series $2, \tfrac{1}{2}, \tfrac{1}{8}, \tfrac{1}{32}, \tfrac{1}{128},$

and $$s = \frac{rl - a}{r - 1}$$

$$= \frac{\tfrac{1}{4}(\tfrac{1}{128}) - 2}{\tfrac{1}{4} - 1}$$

$$= 2\tfrac{85}{128}.$$

The student must use his ingenuity in divising short methods.

(5) Find 61.818181 + to infinity. As the decimal makes a series, we find its sum to infinity, and add it to 61.

In the decimal, we have

$$a = \tfrac{81}{100},$$

$$r = \tfrac{1}{100}.$$

Hence, $$s = \frac{a}{1 - r} = \frac{\tfrac{81}{100}}{\tfrac{99}{100}} = \tfrac{9}{11}.$$

$$\therefore\ 61.818181 = 61\tfrac{9}{11}.$$

The table on the following page will be found useful in solving examples in Geometrical Progression.

As the use of logarithms is necessary in solving when n is required, the solutions for n are not given.

No.	Given.	Required.	Results.
1	$n\ l\ s$	a	$a(s-a)^{n-1}=l(s-l)^{n-1}$.
2	$r\ l\ s$		$a=rl-(r-1)s$.
3	$r\ n\ l$		$a=\frac{l}{r^{n-1}}$.
4	$r\ n\ s$		$a=\frac{(r-1)s}{r^n-1}$.
5	$a\ n\ s$	l	$l(s-l)^{n-1}=a(s-a)^{n-1}$.
6	$a\ r\ n$		$l=ar^{n-1}$.
7	$a\ r\ s$		$l=\frac{a+(r-1)s}{r}$.
8	$n\ r\ s$		$l=\frac{(r-1)sr^{n-1}}{r^n-1}$.
9	$a\ l\ s$	r	$r=\frac{s-a}{s-l}$.
10	$a\ n\ l$		$r=\sqrt[n-1]{\frac{l}{a}}$.
11	$a\ n\ s$		$r^n-\frac{s}{a}r=\frac{a-s}{a}$.
12	$l\ n\ s$		$r^n-\frac{s}{s-l}r^{n-1}=\frac{l}{l-s}$.
13	$a\ l\ n$	s	$s=\frac{\sqrt[n-1]{l^n}-\sqrt[n-1]{a^n}}{\sqrt[n-1]{l}-\sqrt[n-1]{a}}$.
14	$a\ l\ r$		$s=\frac{rl-a}{r-1}$.
15	$a\ n\ r$		$s=\frac{a(r^n-1)}{r-1}$.
16	$l\ n\ r$		$s=\frac{l(r^n-1)}{r^{n-1}(r-1)}$.

EXERCISE III.

1. Find l in $\frac{1}{4}$, $\frac{1}{2}$, 1, $\cdots$ when $n = 7$.

2. Find l in 3, $\frac{3}{2}$, $\frac{3}{4}$, $\frac{3}{8}$, $\cdots$ when $n = 15$.

3. Find l in $\frac{9}{11}$, $1\frac{7}{11}$, $3\frac{3}{11}$, $\cdots$ when $n = 5$.

4. Find l in a, am, am^2, $\cdots$ when $n = 40$.

5. Find l in x, xy, xy^2, $\cdots$ when $n = z$.

6. Find l in $(a + b)$, $(a^2 + 2ab + b^2)$, $\cdots$ when $n = 5$.

7. Find l in $(a + b)$, $(a^2 - b^2)$, $\cdots$ when $n = 4$.

8. Find l in $\frac{1}{2}$, $\frac{a}{2}$, $\frac{a^2}{2}$, when $n = m$.

9. Find l in $(a + b)$, $(am + bm)$, $m^2(a + b)$, when $n = x + y$.

10. Find s in $3 + 6 + 12$, $\cdots$ when $n = 9$.

11. Find s in $\frac{1}{2} + \frac{1}{6} + \frac{1}{18}$, $\cdots$ when $n = 5$.

12. Find s in $\frac{1}{5} + \frac{1}{15} + \frac{1}{45}$, $\cdots$ when $n = 8$.

13. Find s in $\frac{1}{a} + \frac{m}{a} + \frac{m^2}{a}$, $\cdots$ when $n = 7$.

14. Find s in $8\frac{1}{2} + 17 + 34$, $\cdots$ when $n = 8$.

15. Find s in $(a + x)$, $(a + x)^2$, $(a + x)^3$, when $n = 4$.

16. Find s in $-\frac{1}{2} + 4 - 32$, when $n = 7$.

17. Find s in $a^n - a^{2n} + a^{3n}$, when $n = m$.

18. The sum of 5 numbers in geometrical progression is $\frac{31}{32}$ and the first term is $\frac{1}{2}$. Find r and l.

19. Find s in a series of 5 terms in which the third term is $\frac{7}{18}$ and the last term $\frac{7}{162}$.

20. In a series of 5 terms the product is 32 and the last term is 8. What is the series?

21. If $a = 6$, $r = \frac{3}{4}$, and $n = 6$. Find s.

22. If $l = 2047\frac{7}{8}$, $r = 7$, $n = 7$. Find s.

23. Find two numbers whose geometrical mean is 3 and whose sum is $12\frac{3}{4}$.

24. Find two numbers whose difference is 6 and whose arithmetical mean exceeds the geometrical mean by 1.

25. In a series of 8 terms $s = 8$ and $l = 1$. What is the series?

26. Two men, A and B, arrange to play at draughts, as follows: To play as many games as there are squares on the board, to bet 1 cent on the first game, 2 on the second, 4 on the third, and so on. What is the amount of money involved by adding together the winnings? A wins the first 63 games and B the 64th game. Who is the gainer?

27. If daily, in a school, one boy leads another astray, and each boy led astray teaches another to do wrong, what is the number of boys led astray at the end of 6 days? At the end of 30 days?

28. A certain bicyclist can ride a mile in three minutes. He finds by proper care and training he can reduce his time 30 seconds the first year, 15 seconds the second year, $7\frac{1}{2}$ seconds the third, and so on as long as he wishes to train. How fast can he ride at the end of four years? Would he ever be able to make a mile in two minutes?

29. A German Count, playing at Monte Carlo at some game of hazard, bet 20 francs on the first game, and doubled his bet on the following games as long as he continued to lose. At the end of the sixth game, not having won, the count was obliged to stop for want of money. How much did he lose?

30. A railroad train, A, starts 10 miles ahead of another, B. If B goes 10 miles while A goes one mile, at what distance from the starting point will they meet?

31. I have a sum of $1200 which I put out at compound interest at 4%. What will it amount to after six years?

32. Find the sum to infinity in the following: 3.333; 4.666.

33. $a=1,\ r=\frac{1}{2},\ n=\infty$; $a=11,\ r=\frac{2}{3},\ n=\infty$.

34. $a=1,\ r=-\frac{1}{2},\ n=\infty$.

35. 10.8686; 19.544; 5.111.

36. $\frac{1}{2}+\frac{1}{8}+\frac{1}{32}\cdots$, $1+\frac{2}{3}+\frac{4}{9}\cdots$, $3+2+\frac{4}{3}\cdots$.

HARMONICAL PROGRESSION.

342. A series of quantities is said to be in *Harmonical Progression* when their reciprocals are in Arithmetical Progression.

Thus, the following series are in Harmonical Progression:

$$1,\ \tfrac{1}{3},\ \tfrac{1}{5},\ \tfrac{1}{7},\ \tfrac{1}{9},\cdots \tag{1}$$

and

$$\tfrac{1}{9},\ \tfrac{4}{33},\ \tfrac{2}{15},\ \tfrac{4}{27},\ \tfrac{1}{6},\cdots. \tag{2}$$

The reciprocals of the above give

$$1,\ 3,\ 5,\ 7,\ 9,\cdots \tag{1}$$

and

$$9,\ 8\tfrac{1}{4},\ 7\tfrac{1}{2},\ 6\tfrac{3}{4},\ 6,\cdots, \tag{2}$$

which are in Arithmetical Progression.

343. The following may be regarded as the general form of Harmonical Progression:

$$\frac{1}{a},\ \frac{1}{a+d},\ \frac{1}{a+2d},\ \frac{1}{a+3d},\cdots\frac{1}{a+(n-1)d},$$

because the reciprocals are in Arithmetical Progression.

Thus it is readily seen from the conditions expressed that all problems in Harmonical Progression may be solved by taking the reciprocals of the quantities involved and proceeding as in Arithmetical Progression.

344. If there be three consecutive terms, *a*, *b*, *c*, in Harmonical Progression, we have by definition,

$$\frac{1}{b}-\frac{1}{a}=\frac{1}{c}-\frac{1}{b}.$$

$$\therefore\ \frac{a-b}{ab}=\frac{b-c}{bc},$$

$$\therefore\ a-b:b-c::a:c.$$

345. That is, *if three quantities are in Harmonical Progression, the first minus the second is to the second minus the third as the first is to the third.*

This relation is sometimes taken as the definition of Harmonical Progression.

346. The Harmonical Mean. If three terms are in Harmonical Progression, the middle term is the mean between the other two.

347. If a, b, and c are in Harmonical Progression, b is the mean between a and c.

Hence, we have by definition:

$$\frac{1}{b}-\frac{1}{a}=\frac{1}{c}-\frac{1}{b}. \qquad \therefore \frac{2}{b}=\frac{1}{c}+\frac{1}{a}, \qquad \therefore b=\frac{2ac}{a+c}.$$

348. Therefore, *the Harmonical Mean between two quantities is twice their product divided by their sum.*

349. Between any two given quantities any number of means may be inserted.

350. *To insert* n *harmonical means between any two quantities, a and b, insert* n *arithmetical means between $\frac{1}{a}$ and $\frac{1}{b}$, and take their reciprocals.*

351. If b denote the common difference, the Arithmetical Progression will be:

$$\frac{1}{a}, \frac{1}{a}+b \cdots \frac{1}{a}+nb, \frac{1}{c}.$$

and $$\frac{1}{c}=\frac{1}{a}+(n+1)b, \quad \therefore b=\frac{a-c}{(n+1)ac}.$$

That is,

$$\frac{1}{a}, \frac{c(n+1)+a-c}{ac(n+1)}, \cdots \frac{c(n+1)+n(a-c)}{ac(n+1)}, \frac{1}{c}.$$

Therefore the Harmonical Progression will be:

$$a,\ \frac{ac(n+1)}{c(n+1)+a-c},\ \cdots \frac{ac(n+1)}{c(n+1)+n(a-c)},\ c.$$

352. If A, G, H, be the arithmetical, geometrical, and harmonical means, we have

$$A=\frac{a+b}{2},$$

$$G=\sqrt{ab},$$

$$H=\frac{2\,ab}{a+b}.$$

353. From which, $AH=\dfrac{a+b}{2}\times\dfrac{2\,ab}{a+b}=ab=G^2,$

$$\therefore\ G=\sqrt{AH}.$$

That is, G is the geometrical mean between A and H.

354. The quantities a, b, c, are in Arithmetical, Geometrical, or Harmonical Progression, as

$$\frac{a-b}{b-c}=\frac{a}{a},\ \frac{a}{b},\ \text{or}\ \frac{a}{c}.$$

The first gives $b=\frac{1}{2}(a+c)$.
The second gives $b^2=ac$.
The third is obvious from definition.

EXERCISE 112.

1. Insert 8 harmonical means between 1 and $\frac{1}{10}$.
2. Continue the series $2+\frac{2}{3}+\frac{2}{5}$ to 5 terms.
3. Find the 6th term in $\frac{3}{2}+\frac{3}{4}+2+\frac{3}{8}$.
4. Find the 7th term in $\frac{1}{3}+\frac{4}{11}+\frac{2}{5}+\frac{4}{9}$.
5. Find the sum of $\dfrac{1}{a}+\dfrac{1}{a+b}+\dfrac{1}{a+2\,b}$ to 5 terms.

6. Find the last term of $\frac{2}{9}+\frac{8}{37}+\frac{4}{19}+\frac{8}{39}+\frac{1}{5}$ to 10 terms.

7. The first two terms of a Harmonical Progression are 4 and 5. What will the 10th term be?

8. The difference between the arithmetical and harmonical means of two numbers is $\frac{1}{2}$, and one of the numbers is three times the other. What are the series, H. P. and A. P.?

9. Find the nth term of a Harmonical Progression of which x and y are respectively the first and second terms.

10. Find two numbers whose harmonical mean is $3\frac{1}{5}$ and whose difference is 6.

11. Insert 10 harmonical means between $\frac{8}{15}$ and $\frac{8}{4}$.

12. The harmonical mean between two numbers equals $3\frac{1}{5}$; the arithmetical mean equals 5. What are the numbers?

13. The geometrical mean between two numbers is 10; the harmonical mean is 8. What are the numbers?

14. Find the harmonical mean of two numbers whose geometrical mean is 20 and whose arithmetical mean is 25.

15. The arithmetical mean between two numbers equals 0; the geometrical mean equals $\sqrt{-1}$; the harmonical mean equals ∞. What are the numbers?

CHAPTER XXXI.

INDETERMINATE COEFFICIENTS.

355. Convergent and Divergent Series. If the sum of the terms of a series approaches a fixed finite value when the number of terms is indefinitely increased, the series is **Convergent.**

356. If the sum of the terms of a series increases without end, or oscillates in value without approaching a fixed finite value when the number of terms is indefinitely increased, the series is **Divergent.**

357. The theorem for expanding expressions into series is called the **Theorem of Indeterminate Coefficients,** and is as follows:

358. *If the series* $A + Bx + Cx^2 + Dx^3 + \cdots$ *is always equal to the series* $A' + B'x + C'x^2 + D'x^3 + \cdots$, *when* x *has values which make both convergent, the coefficient of the like powers of* x *will be equal. That is,* $A = A'$, $B = B'$, *etc.*

Now if the equation

$$A + Bx + Cx^2 + Dx^3 + \cdots = A' + B'x + C'x^2 + D'x^3 + \cdots$$

is true, we have by transposition

$$A - A' = (B' - B)x + (C' - C)x^2 + (D' - D)x^3,$$

and if x is made less than any assignable quantity, $A - A'$ will be less than any assignable quantity. That is,

$$A - A' = 0, \text{ or } A = A'.$$

Hence, $Bx + Cx^2 + Dx^3 + \cdots = B'x + C'x^2 + D'x^3 + \cdots$.

Dividing by x and transposing, we have

$$B - B' = (C' - C)x + (D' - D)x^2,$$

and by the same proof as above,

$$B - B' = 0, \text{ or } B = B',$$

and so on.

Hence, as a finite series is always convergent, it follows that the coefficient of the like powers of x in the two finite series

$$A + Bx + Cx^2 + Dx^3 + \cdots = A' + B'x + C'x^2 + D'x^3 + \cdots$$

are equal if the two series are equal for every value of x.

359. Performing the operation indicated in the fraction $\frac{1}{1-x}$, we obtain the infinite series $1 + x + x^2 + x^3 + \cdots$. For all values, however, this series is not equal to the fraction.

360. *When x is numerically less than 1, the series is equal to the fraction, and is convergent.*

361. *When x is numerically greater than 1, the series is not equal to the fraction, and is divergent.*

Put $x = \frac{1}{3}$, and the value of the fraction becomes

$$1 + \tfrac{1}{3} + \tfrac{1}{9} + \tfrac{1}{27} + \tfrac{1}{81} + \cdots.$$

Here the value of the sum of the first four terms gives $1\frac{13}{27}$; of the first five terms $1\frac{40}{81}$.

Continuing this process further, we find that the successive sums will approach but never reach $1\frac{1}{2}$.

Put $x = 3$, and the value of the fraction becomes

$$1 + 3 + 9 + 27 + 81 + \cdots.$$

Here the value of the sum of the first four terms is 40; of the first five terms, 121. Continuing this process further, we find that the successive sums are larger and larger, and the fraction and series are not equal.

Put $x = 1$, and we have

$$1 + 1 + 1 + 1 + \cdots.$$

Hence the value of the series can be made to exceed any assigned value, and the series is not equal to the fraction.

Hence, when $x = 1$, the series is divergent.

Put $x = -1$, and we have

$$1 - 1 + 1 - 1 + 1 - 1 + \cdots.$$

362. Hence, the value of the successive terms of the series is either 1 or 0, as the number of the terms is odd or even, and the series is not equal to the fraction.

363. *From the above reasoning, it is evident that an infinite series cannot be used for the purposes of demonstration, unless it is convergent.* That is, when x lies between $+1$ and -1.

364. *If the value of each term of a series, which is alternately positive and negative, is less than the preceding term, the series is convergent.*

For let the series be

$$v_1 - v_2 + v_3 - v_4 + v_5 - v_6 + \cdots.$$

This series may be written

$$(v_1 - v_2) + (v_3 - v_4) + (v_5 - v_6), \qquad (1)$$

or

$$v_1 - (v_2 - v_3) - (v_4 - v_5). \qquad (2)$$

Now by hypothesis each of the expressions $v_1 - v_2$, $v_3 - v_4$ is positive as is evident from (1); and it is also evident from (2) that the sum of any number of terms is less than v_1.

Take for example the series

$$1 - \tfrac{1}{2} + \tfrac{1}{4} - \tfrac{1}{8} + \tfrac{1}{16} - \tfrac{1}{32}.$$

The sum of the first four terms is $\frac{5}{8}$; of the first six terms, $\frac{21}{32}$. And the series approaches, but never reaches, $\frac{2}{3}$ as its limit.

EXPANSION OF FRACTIONS BY MEANS OF INDETERMINATE COEFFICIENTS.

Expand:

(1) $\dfrac{1-x}{1-x+x^2}$ in ascending powers of x.

Assume $\dfrac{1-x}{1-x+x^2}=A+Bx+Cx^2+Dx^3+\cdots.$

Clearing of fractions, we have

$$1-x=\left\{\begin{array}{r|r|r|l} A+B & x+C & x^2+D & x^3\cdots \\ -A & -B & -C & \\ & +A & +B & . \end{array}\right.$$

$$\therefore\ 1-x=A+(B-A)x+(C-B+A)x^2$$
$$+(D-C+B)x^3+\cdots.$$

$$\therefore\ A=1,$$
$$B-A=-1,$$
$$C-B+A=0,$$
$$D-C+B=0.$$

Solving these equations, we have

$$A=1,$$
$$B=0,$$
$$C=-1,$$
$$D=-1.$$

$$\therefore\ \frac{1-x}{1-x+x^2}=1-x^2-x^3\cdots,\text{ etc.}$$

365. For the purposes of demonstration it is **necessary** and **sufficient** that the assumed equation in its simplified form shall have all the powers of x in the right-hand member that are found in the left-hand member.

The coefficient of any power of x in the right-hand member, not in the left-hand member, will vanish — as shown in the above example — and the method still applies.

Any power of x in the left-hand member not in the right-hand member leads to an absurdity.

If we assume in the example given above

$$\frac{1-x}{1-x+x^2}=Ax+Bx^2+Cx^3+Dx^4+\cdots,$$

we would have no term in the right-hand member corresponding to one in the left-hand; hence in equating, $1=0$.

EXERCISE 113.

Expand to five terms in ascending powers of x:

1. $\dfrac{1}{3-4x}$.

2. $\dfrac{5+x}{8+3x}$.

3. $\dfrac{1+x}{1+x+x^2}$.

4. $\dfrac{1}{3+2x+5x^2}$.

5. $\dfrac{5x+3x^2}{2+3x+4x^2}$.

6. $\dfrac{1}{1+3x+x^2}$.

7. $\dfrac{3x}{1-x+2x^2}$.

8. $\dfrac{x-3x^2+x^3}{1+3x+x^2}$.

9. $\dfrac{1+3x^2}{1-3x^2+5x^3}$.

10. $\dfrac{1+4x}{1+x+x^2}$.

PARTIAL FRACTIONS.

366. The Theorem of Indeterminate Coefficients enables us to express a given fraction as the sum of two or more **partial fractions** whose denominators are factors of the denominator of the given fraction, and whose numerators contain no power of x, when the factors of the denominator are of the first degree of x, and the numerator is of a lower degree than the denominator.

367. In the separation of fractions into partial fractions by the above system of indeterminate coefficients, three cases may occur:

I. *When the factors of the denominator are unequal.*
II. *When the factors of the denominator are equal.*
III. *When some of the factors of the denominator are equal.*

Case I.

Separate into partial fractions:

(1) $\dfrac{3+5x}{x^3+11x^2+28x}$.

Resolving x^3+11x^2+28x into fractions, we have,

$$x(x+7)(x+4).$$

Hence, $$\frac{3+5x}{x(x+7)(x+4)}=\frac{A}{x}+\frac{B}{x+4}+\frac{C}{x+7}. \tag{1}$$

Clearing of fractions, we have

$$\begin{aligned}3+5x &= A(x^2+11x+28)+B(x^2+7x)+C(x^2+4x)\\ &= Ax^2+11Ax+28A+Bx^2+7Bx+Cx^2+4Cx.\end{aligned}$$

Hence, $$\begin{aligned}28A&=3,\\ 11A+7B+4C&=5,\\ A+B+C&=0.\\ \therefore\ A&=\tfrac{3}{28},\\ B&=\tfrac{17}{12},\\ C&=-\tfrac{32}{21}.\end{aligned}$$

Substituting the values thus obtained for A, B, and C in (1), we have

$$\begin{aligned}\frac{3+5x}{x^3+11x^2+28x}&=\frac{\frac{3}{28}}{x}+\frac{\frac{17}{12}}{x+4}+\frac{-\frac{32}{21}}{x+7}\\ &=\frac{3}{28x}+\frac{17}{12x+48}-\frac{32}{21x+147}.\end{aligned}$$

CASE II.

Separate into partial fractions:

(2) $\dfrac{x^2+3x+1}{(x+1)^3}$.

Separating according to Case I., we have

$$\frac{x^2+3x+1}{(x+1)^3}=\frac{A}{x+1}+\frac{B}{x+1}+\frac{C}{x+1}$$
$$=\frac{A+B+C}{x+1},$$

but this is plainly impossible.

Substituting $y-1$ for x in the given fraction, we have

$$\frac{(y-1)^2+3(y-1)+1}{\{(y-1)+1\}^3}=\frac{y^2+y-1}{y^3}$$
$$=\frac{1}{y}+\frac{1}{y^2}-\frac{1}{y^3}.$$

Replacing y by $x+1$, we have

$$\frac{1}{x+1}+\frac{1}{(x+1)^2}-\frac{1}{(x+1)^3}.$$

368. Hence, *the given fraction can be resolved into partial fractions whose numerators are independent of x and whose denominators are respectively* $(x+1)$, $(x+1)^2$, $(x+1)^3$.

Hence,
$$\frac{x^2+3x+1}{(x+1)^3}=\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{(x+1)^3}.$$

Clearing of fractions, we have

$$x^2+3x+1=A(x+1)^2+B(x+1)+C$$
$$=Ax^2+2Ax+A+Bx+B+C.$$
$$\therefore\ A+B+C=1,$$

$$2A + B = 3,$$
$$A = 1.$$
$$\therefore\ A = 1,$$
$$B = 1,$$
$$C = -1.$$

And by substitution we have

$$\frac{x^2 + 3x + 1}{(x+1)^3} = \frac{1}{x+1} + \frac{1}{(x+1)^2} - \frac{1}{(x+1)^3}.$$

CASE III.

Separate into partial fractions

$$\frac{3x + 4}{x(x+5)^2}.$$

Proceeding according to Case I. and Case II., we have

$$3x + 4 = \frac{A}{x} + \frac{B}{x+5} + \frac{C}{(x+5)^2}.$$

Clearing of fractions, we have

$$A(x+5)^2 + B(x^2 + 5x) + Cx$$
$$= Ax^2 + 10Ax + 25A + Bx^2 + 5Bx + Cx.$$
$$\therefore\ 25A = 4,$$
$$10A + 5B + C = 3,$$
$$A + B = 0.$$
$$\therefore\ A = \tfrac{4}{25},$$
$$B = -\tfrac{4}{25},$$
$$C = \tfrac{11}{5}.$$

Substituting these values, we have

$$\frac{3x + 4}{x(x+5)^2} = \frac{4}{25x} - \frac{4}{25(x+5)} + \frac{11}{5(x+5)^2}.$$

EXERCISE 114.

Separate the following into partial fractions:

1. $\dfrac{1}{x(x+1)^2}$.
2. $\dfrac{1+2x}{x^2+3x+2}$.
3. $\dfrac{1+2x+x^2}{2-3x+x^2}$.
4. $\dfrac{4+5x+6x^2}{x^2+15x+56}$.
5. $\dfrac{x^3+2x^2+1}{x^4-2x^2+1}$.
6. $\dfrac{1}{(x-1)^3}$.
7. $\dfrac{3+7x}{(1+x)^3}$.
8. $\dfrac{3+x+6x^2}{(x+1)^2(x-1)^2}$.
9. $\dfrac{8+x+9x^2}{x^3(x+3)^3}$.
10. $\dfrac{3+3x^2}{(x-1)(x+2)(x-3)^2}$.
11. $\dfrac{3x-14}{(x^2-9)(x^2-25)}$.
12. $\dfrac{7x+11}{(x^2+5x+4)(x^2-5x+4)}$.
13. $\dfrac{15+4x+5x^2}{(x+4)^2(x-5)^2}$.
14. $\dfrac{x^2}{(3x-1)(4x+6)(3x-7)}$.

369. If the denominator of the given fraction contains factors of the first and factors of the second degree, the numerators of the quadratic factors are not independent of x, and should be expressed in the form $Ax+B$.

When, however, the factors are of the second degree and all equal, Case II. applies.

Separate into partial fractions:

$$\frac{2}{1+x^3}.$$

Assume $$\frac{2}{1+x^3}=\frac{A}{1+x}+\frac{Bx+C}{1-x+x^2}.$$

Clearing of fractions,

$$\begin{aligned}2&=A(1-x+x^2)+(Bx+C)(1+x)\\&=A-Ax+Ax^2+Bx+Bx^2+C+Cx.\end{aligned}$$

Hence,

$$\begin{aligned} A + C &= 2, \\ -A + B + C &= 0, \\ A + B &= 0. \\ \therefore\ A &= \tfrac{2}{3}, \\ B &= -\tfrac{2}{3}, \\ C &= \tfrac{4}{3}. \end{aligned}$$

Substituting these values of A, B, and C, we have

$$\frac{2}{1+x^3} = \frac{2}{3(1+x)} - \frac{2x-4}{3(1-x+x^2)}.$$

EXERCISE 115.

Separate the following into partial fractions:

1. $\dfrac{2x^2+x-1}{1-x^3}$.
2. $\dfrac{x^2-2x+5}{x^4-81}$.
3. $\dfrac{5x^2+3x-6}{x^4+x^2+1}$.
4. $\dfrac{3x^2-1}{x^4+4}$.
5. $\dfrac{9x^2-5x+2}{x^4+9x^2+14}$.
6. $\dfrac{3x^2+2x+1}{x^3+1}$.

CHAPTER XXXII.

BINOMIAL THEOREM.

370. The formula by which any power of a binomial may be expanded into a series is called the **Binomial Theorem.** This theorem was first explained by Newton.

371. Positive Integral Exponents. By actual multiplication we obtain the following:

$$(a+b)^2 = a^2 + 2ab + b^2,$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3,$$
$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

372. In the above we may note the following laws:

I. *The number of terms is one greater than the index of the binomial.*

II. *The exponent of* a *in the first term is equal to the index, and decreases* 1 *in each succeeding term.*

III. *The exponent of* b *in the second term is* 1, *and increases* 1 *in each succeeding term.*

IV. *The coefficient of the first term is* 1, *and of the second term the index of the binomial.*

V. *Also, if the coefficient of any term is multiplied by the exponent of* a *in that term, and the result divided by the number of the term, the quotient will be the coefficient of the next following term.*

From the above we know that the law holds good for the second, third, and fourth power. Let us assume that it holds good for any positive integer; *i.e.* that it holds good for the nth power.

Assume
$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 \cdots. \qquad (1)$$

Now, multiplying both members by $a+b$, we have,

$$(a+b)^{n+1} = a^{n+1} + (n+1)a^n b + \frac{(n+1)n}{1\cdot 2}a^{n-1}b^2 + \frac{(n+1)n(n-1)}{1\cdot 2\cdot 3}a^{n-2}b^3 + \cdots. \qquad (2)$$

Now, substituting $n+1$ for n in (1), we have,

$$(a+b)^{n+1} = a^{n+1} + (n+1)a^n b + \frac{(n+1)n}{1\cdot 2}a^{n-1}b^2 + \frac{(n+1)n(n-1)}{1\cdot 2\cdot 3}a^{n-2}b^3 + \cdots. \qquad (3)$$

Equation (3) is the same as equation (2).

Hence, if the laws hold good for the nth power, they also hold for the $(n+1)$th power, and for any positive integral exponent.

But holding good for the second, third, and fourth power, they hold for the fifth, sixth, seventh, and nth power and any positive integral exponent.

The expansion of $(a+b)^n$ is a finite series when n is a positive integer; for, finally, a term will contain $n-n$. This term will vanish as will the succeeding terms for like reason.

If b is negative, terms containing odd powers of b will be negative, while those containing even powers will be positive.

373. The expressions $1\cdot 2\cdot 3$, $1\cdot 2\cdot 3\cdot 4$, $1\cdot 2\cdot 3\cdot 4 \cdots n$ are generally written $\underline{|3}$, $\underline{|4}$, $\underline{|n}$, and are read "factorial 3," "factorial 4," and "factorial n."

Expand $(a+b)^6$.

$$a^6 + 6a^5b + \frac{6 \cdot 5}{1 \cdot 2}a^4b^2 + \frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3}a^3b^3$$

$$+ \frac{6 \cdot 5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3 \cdot 4}a^2b^4 + \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}ab^5$$

$$+ \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}b^6,$$

which equals

$$a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6.$$

It will be noticed in the above expansion that the terms equally distant from the ends have the same exponent. Also that the coefficient of the middle term (or the two middle terms if the power is odd) is the highest.

374. *If the terms of an expansion contain Arabic numerals, or have exponents other than unity, it will be found convenient to enclose them in parentheses before expansion.*

(1) Expand $(2x+3y)^4$.

$$(2x)^4 + 4(2x)^3(3y) + \frac{4 \cdot 3}{1 \cdot 2}(2x)^2(3y)^2$$

$$+ \frac{4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3}(2x)(3y)^3 + \frac{4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 . 4}(3y)^4,$$

which reduced gives the following:

$$16x^4 + 96x^3y + 216x^2y^2 + 216xy^3 + 81y^4.$$

(2) Expand $(x^{-\frac{1}{2}} + y^{\frac{2}{3}})^3$.

$$(x^{-\frac{1}{2}})^3 + 3(x^{-\frac{1}{2}})^2(y^{\frac{2}{3}}) + \frac{3 \cdot 2}{1 \cdot 2}(x^{-\frac{1}{2}})(y^{\frac{2}{3}})^2 + \frac{3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3}(y^{\frac{2}{3}})^3,$$

which reduced gives the following:

$$x^{-\frac{3}{2}} + 3x^{-1}y^{\frac{2}{3}} + 3x^{-\frac{1}{2}}y^{\frac{4}{3}} + y^2.$$

(3) Expand $(2a-\sqrt{b})^4$.

$$(2a)^4+4(2a)^3(-\sqrt{b})+\frac{4\cdot 3}{1\cdot 2}(2a)^2(-\sqrt{b})^2$$
$$+\frac{4\cdot 3\cdot 2}{1\cdot 2\cdot 3}(2a)(-\sqrt{b})^3+\frac{4\cdot 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\cdot 4}(-\sqrt{b})^4,$$

which reduced gives the following:

$$16a^4-32a^3\sqrt{b}+24a^2b-8ab\sqrt{b}+b^2.$$

EXERCISE 116.

Expand the following:

1. $(a+b)^7$.
2. $(2a-b)^3$.
3. $(2x+y^2)^4$.
4. $(5x-3y^3)^3$.
5. $(x^{-\frac{1}{2}}+2y^{\frac{2}{5}})^4$.
6. $(x^{2m}+1)^5$.
7. $(a^3+3\sqrt{b})^6$.
8. $\left(\frac{a}{2}+\frac{1}{a}\right)^7$.
9. $(x^{m+1}-y^{n-1})^3$.
10. $(a^{-1}b^{\frac{1}{2}}+c^{\frac{3}{4}})^4$.
11. $\left(\frac{\sqrt{x}}{\sqrt{y}}+1\right)^3$.
12. $(\frac{1}{2}x^{-2}+\frac{3}{4}y^{-3})^4$.

Regarding two terms of a trinomial as one, and enclosing the same in a parenthesis, the trinomial may be expanded by the binomial formula.

(4) Expand $(a+b-c)^3$.

Arrange as follows:

$$[(a+b)-c]^3,$$

and we have, by expansion,

$$(a+b)^3+3(a+b)^2(-c)+3(a+b)(-c)^2+(-c)^3,$$

which reduced gives the following:

$$a^3+3a^2b+3ab^2+b^3-3a^2c-6abc-3b^2c+3ac^2+3bc^2-c^3.$$

Expand the following:

1. $(a^2 - b + c)^3$.
2. $(2x - 3y + 1)^4$.
3. $(x^{-2} + 1 - y^3)^5$.
4. $(1 - a + a^2)^5$.
5. $\left(\frac{a}{2} + \frac{b}{3} + \frac{c}{4}\right)^3$.
6. $(2x - 3y + z)^4$.

375. To find any required term.

In the expansion of $(a + b)^n$, the following laws will hold when it is required to find any term:

I. *The exponent of b is 1 less than the number of the term, and the exponent of a is equal to n minus the exponent of b.*

If it be required to find the rth term, the exponent of b is $r - 1$, *and the exponent of a,* $n - (r - 1)$ *or* $n - r + 1$.

II. *In arranging the coefficient, the last factor of the numerator is 1 greater than the exponent of a, and the last factor of the denominator is equal to the exponent of b.*

Thus if the rth term is required, the last factor of the numerator will be $n - r + 2$, *and the last factor of the denominator will be* $r - 1$.

From the above, the *r*th term may be written as follows:

$$\frac{n(n-1)(n-2)(n-3)\cdots(n-r+2)}{1\cdot 2\cdot 3\cdot 4\cdots(r-1)}a^{n-r+1}b^{r-1}.$$

Find the 6th term of $(a + b)^{10}$.

Substituting in the above formula, we have,

$$\frac{10(10-1)(10-2)(10-3)(10-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5}a^{10-6+1}b^{6-1},$$

which reduced gives the following:

$$252\,a^5b^5.$$

EXERCISE 117.

Find the required terms of the following:

1. The 5th term of $(a + b)^8$.
2. The 7th term of $(1 + x)^{11}$.

3. The 4th term of $(x+2y)^{15}$.

4. The 10th term of $(a+3b)^{12}$.

5. The 8th term of $(\sqrt{x}+1)^{15}$.

376. What has been shown of the sign of any term, and the value of coefficients equally distant from the beginning and end of the expansion, applies in finding any required term.

6. Find the 5th term of $(x-y)^9$.

7. Find the 8th term of $\left(x^{-2}-\frac{y}{2}\right)^{12}$.

8. Find the 50th term of $(a-b)^{52}$.

9. Find the 99th term of $(a+b)^{101}$.

10. Find the middle term of $(x-3y)^{16}$.

FRACTIONAL AND NEGATIVE EXPONENTS.

377. It has already been shown that when n is a positive integer, we have,

$$(1+x)^n = 1 + nx^n + \frac{n(n-1)}{1 \cdot 2}x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}x^3 + \cdots.$$

Suppose for the moment that $n = \frac{p}{q}$.

Now, as this proof has for its object the expansion of $(1+x)^n$ into a convergent series, we may assume that,

$$(1+x)^n = A + Bx + Cx^2 + Dx^3 + \cdots.$$

Now, substituting $\frac{p}{q}$ for n,

$$(1+x)^{\frac{p}{q}} = A + Bx + Cx^2 + Dx^3 + \cdots,$$

which equals, when each member is raised to the qth power,

$$(1+x)^p = (A + Bx + Cx^2 + Dx^3 + \cdots)^q. \qquad (1)$$

Expanding both members, we have,

$$1+px+\frac{p(p-1)}{1\cdot 2}x^2+\cdots=$$

$$A^q+qA^{q-1}Bx+\frac{q(q-1)}{1\cdot 2}(A^{q-2}B^2+qA^{q-1}C)x^2+\cdots.$$

Equating the coefficient of the corresponding term, we have,

$$A^q=1. \quad \therefore\ A=1.$$

$$qA^{q-1}B=p. \quad \therefore\ B=\frac{p}{q}.$$

Hence, substituting in (1) and extracting the qth root of both members of the equation, we have,

$$(1+x)^{\frac{p}{q}}=1+\frac{p}{q}x+Cx^2+Dx^3+\cdots. \qquad (2)$$

Suppose, again, that n is negative, integral, or fractional.

Let $n=-m$.

Then we have,

$$(1+x)^n=(1+x)^{-m}=\frac{1}{(1+x)^m},$$

and from (2) we have,

$$\frac{1}{(1+x)^m}=\frac{1}{1+mx+Cx^2+Dx^3+\cdots},$$

which gives, by division,

$$(1+x)^{-m}=1-mx+Cx^2+Dx^3+\cdots. \qquad (3)$$

It therefore appears, from (2) and (3), that whether the index of the power be positive, negative, or fractional, the form of expansion is as follows:

$$(1+x)^n=1+nx+Cx^2+Dx^3+\cdots. \qquad (4)$$

378. To find the other coefficients in this expression.

Put $x+z$ for x in (4), and we have,

$$[1+(x+z)]^n \text{ or } [(1+z)+x]^n.$$

Expand each of these expressions, and we have,

$$\begin{aligned}[1+(x+z)]^n &= 1+n(x+z)+C(x+z)^2+D(x+z)^3+\cdots \\ &= 1+nx+nz+Cx^2+2\,Cxz+Cz^2 \\ &\quad +Dx^3+3\,Dx^2z+\cdots \\ &= 1+nx+Cx^2+Dx^3+\cdots+nz \\ &\quad +2\,Cxz+3\,Dx^2z+\cdots. \qquad (5)\end{aligned}$$

Again, $[(1+z)+x]^n$

$$\begin{aligned} &= (1+z)^n\left(1+\frac{x}{1+z}\right)^n \\ &= (1+z)^n\left(1+\frac{nx}{1+z}+\frac{Cx^2}{(1+z)^2}+\cdots\right) \\ &= (1+z)^n+n(1+z)^{n-1}x+C(1+z)^{n-2}x^2+\cdots \\ &= 1+nx+Cx^2+\cdots \\ &\quad +nz+n(n-1)zx+C(n-2)zx^2+\cdots. \qquad (6)\end{aligned}$$

Now $(1+x+z)^n$ and $(1+z+x)^n$ are same expressions. The coefficients, therefore, of the corresponding term in (5) and (6) must be same.

Equating these coefficients, for z, zx, zx^2, we have,

$$\begin{aligned} n &= n; \\ 2\,C &= n(n-1); \\ 3\,D &= C(n-2). \\ \therefore\ C &= \frac{n(n-1)}{\lfloor 2}; \end{aligned}$$

$$D=\frac{C(n-2)}{3}=\frac{n(n-1)(n-2)}{\lfloor 3}\cdots.$$

Substituting these values in (4), we have,

$$(1+x)^n=1+nx+\frac{n(n-1)}{\lfloor 2}+\frac{n(n-1)(n-2)}{\lfloor 3}+\cdots,$$

which proves the binomial theorem for all values of n.

379. To find the cube root of a number by means of the binomial theorem.

I. *Separate the given number into two parts, one of which shall be its nearest perfect cube.*

II. *Placing the perfect power as the first term of the binomial, and the remaining part as the second term, expand by means of the binomial formula.*

Find, by means of the binomial theorem, the cube root of 65.

$$\sqrt[3]{65} = (4^3 + 1)^{\frac{1}{3}}.$$

Expanding, we have,

$$\begin{aligned}\sqrt[3]{65} &= (4^3)^{\frac{1}{3}} + \tfrac{1}{3}(4^3)^{-\frac{2}{3}} - \tfrac{1}{9}(4^3)^{-\frac{5}{3}} + \tfrac{5}{81}(4^3)^{-\frac{8}{3}} - \cdots\\ &= 4 + \tfrac{1}{48} - \tfrac{1}{9216} + \tfrac{1}{5357016} - \cdots\\ &= 4 + .02083 - .0001 + .0000001 - \cdots\\ &= 4.0207301 +.\end{aligned}$$

In like manner any root may be found. Thus,

$$\sqrt[5]{31} = (2^5 - 1)^{\frac{1}{5}}.$$

EXERCISE 118.

1. $\sqrt{17}$.
2. $\sqrt{37}$.
3. $\sqrt{24}$.
4. $\sqrt{33}$.
5. $\sqrt[3]{10}$.
6. $\sqrt[3]{25}$.
7. $\sqrt[3]{63}$.
8. $\sqrt[4]{18}$.
9. $\sqrt[4]{80}$.
10. $\sqrt[6]{66}$.
11. $\sqrt[5]{35}$.
12. $\sqrt[3]{342}$.

CHAPTER XXXIII.

LOGARITHMS.

380. If we have given two progressions, the one geometrical beginning with unity, the other arithmetical beginning with zero; *i.e.*,

$$1,\ a,\ a^2,\ a^3,\ a^4, \cdots,$$

$$0,\ b,\ 2b,\ 3b,\ 4b, \cdots,$$

the terms of the arithmetical progression are called the logarithms of the corresponding terms of the geometrical progression, supposing, in general, that the ratio in the geometrical system is greater than unity, and the difference in the arithmetical system is positive. These two progressions constitute the system of logarithms.

If the term a of the geometrical progression is greater than unity, and the term b of the arithmetical progression is positive, both progressions may increase to infinity.

By the above application of the one progression to another, we may find the logarithm of any number which corresponds to any term of the geometrical progression. The rule may be made applicable to all numbers; for, insert a very large number of geometrical means between any two terms of the geometrical progression, and we shall form a new progression, the terms of which will be less removed the one from the other. Inserting again between the two corresponding terms of the arithmetical progression an equal number of means, we shall have a new set of terms, giving exactly the logarithms of the terms inscribed in the geometrical progression, and approximately of other numbers

By the above process, the logarithms of all numbers may be reached either exactly or approximately.

381. The Common System, or Briggs' System of Logarithms, has 10 for its base, and is the only one used for numerical calculation.

Every positive real number may be expressed as a power of 10 exactly or approximately.

Thus,
$$\begin{aligned} 1 &= 10^0; \\ 1000 &= 10^3; \\ 17 &= 10^{1.2304}; \\ 95 &= 10^{1.9777}. \end{aligned}$$

From the above, therefore, it is seen that the logarithm of a number in the *Common System* is simply an **exponent** representing the power to which 10 must be raised to produce the given number.

Thus, the logarithm of $100 = 2$. This indicates that 10 raised to the second power will produce 100.

And, in general, if $10^x = y$, $x = \log y$.

382. By the theory of exponents as already explained, we have,

$$\begin{array}{ll} 10^0 = 1. & 10^{-1} = \frac{1}{10} = .1. \\ 10^1 = 10. & 10^{-2} = \frac{1}{10^2} = .01. \\ 10^2 = 100. & 10^{-3} = \frac{1}{10^3} = .001. \\ 10^3 = 1000. & 10^{-4} = \frac{1}{10^4} = .0001. \end{array}$$

Therefore,

$$\begin{array}{ll} \log 1 = 0. & \log .1 = -1. \\ \log 10 = 1. & \log .01 = -2. \\ \log 100 = 2. & \log .001 = -3. \\ \log 1000 = 3. & \log .0001 = -4. \end{array}$$

383. The logs of .1, .01, .001, .0001 are written, in practice, 9 – 10, 8 – 10, 7 – 10, and 6 – 10, respectively.

384. A negative number has no logarithm.

In working with negative numbers, disregard the minus sign, and attach the proper algebraic sign to the result.

385. From what has been shown above, it is evident that the logarithms of all numbers greater than 1 are positive; and that the logarithms of all numbers between 0 and 1 are negative.

386. The integral part of a logarithm is called the **characteristic**; the decimal part is called the **mantissa.**

Thus, $\log 27 = 1.43136$.

Here 1 is the *characteristic,* and .43136 the *mantissa.*

387. Generally the mantissa only is given in the table of logarithms. For it is evident that all numbers between 1 and 10 will have a logarithm of 0 plus a decimal fraction;

between 10 and 100, 1 plus a decimal;
between 100 and 1000, 2 plus a decimal.

Therefore, for numbers between

1 and 10, the characteristic is 0;
10 and 100, the characteristic is 1;
100 and 1000, the characteristic is 2,

and so on; and, in general, *the characteristic of any number greater than* 1 *is* 1 *less than the number of places on the left of the decimal point.*

Thus, 5 is the characteristic of 346724.97852.

388. Again, the logarithms of all numbers between 1 and .1 lie between 0 and – 1, and therefore are, by Art. 382, decimals with a negative sign.

389. In practice, however, the *mantissa is always positive*, and the logarithms of numbers between

1 and .1 are $9 +$ a decimal $- 10$;
.1 and .01 are $8 +$ a decimal $- 10$;
.01 and .001 are $7 +$ a decimal $- 10$;
.001 and .0001 are $6 +$ a decimal $- 10$,

and so on.

And, in general, the characteristic of a decimal is found by subtracting the number corresponding to the first *significant* place from 10 and placing $- 10$ after the mantissa.

Thus, the logarithm of .4052 is $9 +$ the mantissa $- 10$, and of .00824 is $7 +$ the mantissa $- 10$.

390. The negative part of the logarithm may, however, be placed before the mantissa.

Thus, the $\log 8.0234 - 10$ may be written $\bar{2}.0234$. The negative sign is here placed above the characteristic to show that it alone is affected by it. The meaning in the $\log \bar{2}.0234$ is $-2 + .0234$, from which it is clearly seen that adding and subtracting 10 will give $8.0234 - 10$, a result identical in value but of more convenient form in practice.

SOME FUNDAMENTAL PROPERTIES OF LOGARITHMS.

391. I. *Regarding m as the base of any system, we have* $m^0 = 1$ *and* $m^1 = m$; *that is, the* $\log 1 = 0$ *and the* $\log m = 1$.

392. II. *Again, if* $m > 1$, $m^{-\infty} = \frac{1}{m^\infty} = \frac{1}{\infty} = 0$; *that is, the log of zero is minus infinity when the base is* > 1.

Also if $m < 1$, $m^\infty = 0$; *that is, the log of zero is* ∞ *when the base is* < 1.

393. III. *Again, if* $m^x = a$, *and* $m^y = b$, *we have* $m^{x+y} = ab$. *Hence* $\log ab = x + y$; *that is, the logarithm of the product of two numbers is equal to the sum of the logarithms of its factors.*

394. IV. *Also if* $m^x = a$, *and* $m^y = b$, $m^{x-y} = \frac{a}{b}$. *Hence,* $\log \frac{a}{b} = x - y$; *that is, the logarithm of the quotient of two numbers is equal to the logarithm of the dividend minus the logarithm of the divisor.*

395. V. *Again, if* $m^x = a$, $m^{px} = a^p$, *and* $\log a^p = px = \log a \times p$; *that is, the logarithm of a power of any quantity equals the logarithm of the quantity multiplied by the index of the power.*

396. VI. *Also if* $\sqrt[x]{m^y} = a$, $m^{\frac{y}{x}} = a$, *and* $\log a = \frac{y}{x}$; *that is, the logarithm of the root of any quantity equals the logarithm of the quantity divided by the index of the root.*

Given $\log 2 = .30103$, $\log 3 = .47712$;
$\log 5 = .69897$, $\log 7 = .84510$;
$\log 11 = 1.04139$, $\log 13 = 1.11394$.

EXAMPLES:

1. Find the log of 48.

$$\begin{aligned}\log 48 &= \log (2 \times 2 \times 2 \times 2 \times 3)\\ &= \log 2 + \log 2 + \log 2 + \log 2 + \log 3\\ &= 4 \times \log 2 + \log 3\\ &= 4 \times .30103 + .47712\\ &= 1.20412 + .47712\\ &= 1.68124.\end{aligned}$$

2. Find the log of $\frac{33}{13}$.

$$\begin{aligned}\log \tfrac{33}{13} &= \log\left(\frac{3\times 11}{13}\right)\\ &= (\log 3 + \log 11) - (\log 13)\\ &= (.47712 + 1.04139) - 1.11394\\ &= .40457.\end{aligned}$$

3. Find the log 7^5.

$$\begin{aligned}\log 7^5 &= \log 7 \times 5\\ &= .84510 \times 5\\ &= 4.22550.\end{aligned}$$

4. Find the log $110^{\frac{2}{3}}$.

$$\begin{aligned}\log 110^{\frac{2}{3}} &= \log 110 \times \tfrac{2}{3}\\ &= \log (11 + 10) \times \tfrac{2}{3}\\ &= 1.04139 + 1 \times \tfrac{2}{3}\\ &= 2.04139 \times \tfrac{2}{3}\\ &= 1.36093.\end{aligned}$$

5. Find the log of $\sqrt[7]{2^{11}}$.

$$\begin{aligned}\log \sqrt[7]{2^{11}} &= \log 2^{\frac{11}{7}}\\ &= \log 2 \times \tfrac{11}{7}\\ &= .30103 \times \tfrac{11}{7}\\ &= .47305.\end{aligned}$$

EXERCISE 119.

Given $\log 2 = .3010$, $\log 3 = .4771$, $\log 5 = .6990$, $\log 7 = .8451$; find:

1. $\log 84$.

2. $\log 21$.

3. $\log 315$.

4. $\log 1728$.

5. $\log 3402$.

6. $\log \frac{10}{9}$.

7. $\log .03$.

8. $\log \frac{16}{7}$.

9. $\log \frac{64}{9}$.

10. $\log 42\frac{7}{9}$.

11. $\log 7^5$.

12. $\log 21^2$.

13. $\log 35^{\frac{5}{7}}$.	15. $\log 27^{\frac{1}{2}}$.	17. $\sqrt[5]{7}$.	19. $\sqrt{\frac{2}{7}}$.
14. $\log 16^{\frac{2}{7}}$.	16. $\sqrt[3]{5}$.	18. $\sqrt{\frac{7}{5}}$.	20. $\frac{\sqrt{42}}{9}$.

397. For the same sequence of digits the mantissa does not change.

The logarithm of $3.964 = .59813$.

Now $3964. = 3.964 \times 1000$.

$$\therefore \log 3964. = \log 3.964 + \log 1000$$
$$= .59813 + 3 = 3.59813,$$

and $.003964 = 3.964 \times .001$.

$$\therefore \log .003964 = \log 3.964 + \log .001$$
$$= .59813 + (-3) = 7.59813 - 10,$$

and, in general, if a is any positive or negative integer,

$$\log(10^a \times m) = a \times \log 10 + \log m = a + \log m.$$

398. As any number multiplied by a power of 10 will differ only in position of the decimal, the log of such number will always be the log of the power of 10 plus the log of the number, but the log of the power of 10 is always integral, hence the decimal part of the result will not be changed.

Thus:

$$\begin{aligned}
&\text{log of } 3964. = 3.59813,\\
&\text{log of } 396.4 = 2.59813,\\
&\text{log of } 39.64 = 1.59813,\\
&\text{log of } 3.964 = .59813,\\
&\text{log of } .3964 = \bar{1}.59813 = 9.59813 - 10,\\
&\text{log of } .03964 = \bar{2}.59813 = 8.59813 - 10,\\
&\text{log of } .003964 = \bar{3}.59813 = 7.59813 - 10.
\end{aligned}$$

399. This property, which belongs only to the common system, makes it the only system suited to numerical calculation, and gives it great superiority over other systems; it may be stated as follows:

400. I. *The characteristic depends on the position of the decimal point.*

II. *The mantissa depends on the sequence of digits.*

401. If the logarithm of a number to one base is given, find the logarithm of the same number to another base.

Let a and b be the bases of two systems, and m the number whose $\log x$ to base a is known, then we have,

$$a^x = m.$$

Let y be $\log m$ to base b, then we have,

$$b^y = m.$$

$$\therefore \begin{cases} \log_a m = x, \\ \log_b m = y. \end{cases}$$

$$\therefore\ a^x = b^y, \text{ and } a^{\frac{x}{y}} = b. \quad \therefore\ \log_a b = \frac{x}{y},$$

$$\text{and } y = \frac{x}{\log_a b}; \text{ but } x = \log_a m, \text{ and } y = \log_b m.$$

$$\therefore\ \log_b m = \frac{\log_a m}{\log_a b}. \qquad \text{(I.)}$$

If $m = a$ in (I.), we have

$$\log_b a = \frac{\log_a a}{\log_a b} = \frac{1}{\log_a b}. \quad \therefore\ \log_b a \times \log_a b = 1. \qquad \text{(II.)}$$

EXERCISE 120.

Find the value of the following to the base of 10:

1. $\log_2 12$. 2. $\log_3 14$. 3. $\log_5 29$. 4. $\log_4 51$.

5. Find the logarithm of 324 to the base of 5.

6. Find the logarithm of $\frac{1}{5}$ to the base of 9.

EXPLANATION OF TABLES.

402. In the tables given on pp. 274–275, the mantissa of the logarithms of all numbers between 1 and 1000 are given to four places of decimals.

403. To find the log of a number of one or two significant places, use the column headed 0.

Thus, log 59 = 1.7709, the mantissa of 59 and 590 being identical. The characteristic is 1.

Again, log 9 = .9542, being the same as the mantissa of 900.

NOTE. — The first figure of the mantissa in the table is not repeated, except in the horizontal line in which it first occurs.

404. To find the logarithm of a number of three significant places.

In the column headed No. find the first two significant places.

In the horizontal line with these two significant figures, and in the column headed with the third significant figure, will be found the mantissa of the logarithm required.

Prefix the proper characteristic.

Thus, the log 864 = 2.9365, as .9365 occurs in the horizontal line of 86 and in column headed 4.

As the number has three places to the left of the decimal point, the characteristic is 2.

405. To find a logarithm of a number of more than three figures.

Find the logarithm 36595.

In line 36, column 5, is found .5623, and in line 36, column 6, is found .5635. The difference between these mantissas is .0012. The difference between 36500 and 36600 is 100, and the difference between 36500 and 36595 is 95. Therefore $\frac{95}{100}$ of .0012, or 0011, must be added to .5623 to produce the required mantissa, which is .5634.

Hence log 36595 = 4.5634.

No.	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	414	453	492	531	569	607	645	682	719	755
12	792	828	864	899	934	969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	335	367	399	430
14	461	492	523	553	584	614	644	673	703	732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	279
17	304	330	355	380	405	430	455	480	504	529
18	553	577	601	625	648	672	695	718	742	765
19	788	810	833	856	878	900	923	945	967	989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	222	243	263	284	304	324	345	365	385	404
22	424	444	464	483	502	522	541	560	579	598
23	617	636	655	674	692	711	729	747	766	784
24	802	820	838	856	874	892	909	927	945	962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	183	200	216	232	249	265	281	298
27	314	330	346	362	378	393	409	425	440	456
28	472	487	502	518	533	548	564	579	594	609
29	624	639	654	669	683	698	713	728	742	757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	914	928	942	955	969	983	997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	145	159	172
33	185	198	211	224	237	250	263	276	289	302
34	315	328	340	353	366	378	391	403	416	428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	563	575	587	599	611	623	635	647	658	670
37	682	694	705	717	729	740	752	763	775	786
38	798	809	821	832	843	855	866	877	888	899
39	911	922	933	944	955	966	977	988	999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	128	138	149	160	170	180	191	201	212	222
42	232	243	253	263	274	284	294	304	314	325
43	335	345	355	365	375	385	395	405	415	425
44	435	444	454	464	474	484	493	503	513	522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	628	637	646	656	665	675	684	693	702	712
47	721	730	739	749	758	767	776	785	794	803
48	812	821	830	839	848	857	866	875	884	893
49	902	911	920	928	937	946	955	964	972	981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	093	101	110	118	126	135	143	152
52	160	168	177	185	193	202	210	218	226	235
53	243	251	259	267	275	284	292	300	308	316
54	324	332	340	348	356	364	372	380	388	396
No.	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**

No.	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	482	490	497	505	513	520	528	536	543	551
57	559	566	574	582	589	597	604	612	619	627
58	634	642	649	657	664	672	679	686	694	701
59	709	716	723	731	738	745	752	760	767	774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	853	860	868	875	882	889	896	903	910	917
62	924	931	938	945	952	959	966	973	980	987
63	993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	069	075	082	089	096	102	109	116	122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	195	202	209	215	222	228	235	241	248	254
67	261	267	274	280	287	293	299	306	312	319
68	325	331	338	344	351	357	363	370	376	382
69	388	395	401	407	414	420	426	432	439	445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	513	519	525	531	537	543	549	555	561	567
72	573	579	585	591	597	603	609	615	621	627
73	633	639	645	651	657	663	669	675	681	686
74	692	698	704	710	716	722	727	733	739	745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	808	814	820	825	831	837	842	848	854	859
77	865	871	876	882	887	893	899	904	910	915
78	921	927	932	938	943	949	954	960	965	971
79	976	982	987	993	998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	085	090	096	101	106	112	117	122	128	133
82	138	143	149	154	159	165	170	175	180	186
83	191	196	201	206	212	217	222	227	232	238
84	243	248	253	258	263	269	274	279	284	289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	345	350	355	360	365	370	375	380	385	390
87	395	400	405	410	415	420	425	430	435	440
88	445	450	455	460	465	469	474	479	484	489
89	494	499	504	509	513	518	523	528	533	538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	590	595	600	605	609	614	619	624	628	633
92	638	643	647	652	657	661	666	671	675	680
93	685	689	694	699	703	708	713	717	722	727
94	731	736	741	745	750	754	759	763	768	773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	823	827	832	836	841	845	850	854	859	863
97	868	872	877	881	886	890	894	899	903	908
98	912	917	921	926	930	934	939	943	948	952
99	956	961	965	969	974	978	983	987	991	996
No.	0	1	2	3	4	5	6	7	8	9

The difference between a mantissa of the table and the mantissa next following is called the *tabular difference.*

Find logarithm of .0006725.

The mantissa of $672 = .8274.$

Tabular difference $= .0006,$

$\frac{5}{10} \times .0006 = .0003.$

Hence $.8274 + .0003 = .8277,$ the required mantissa.

The characteristic is -4, or $6 - 10$.

Hence $\log .0006725 = 6.8277 - 10.$

406. To find a number when its logarithm is given.

(1) Given the logarithm 3.4624, find the corresponding number.

.4624 is found in the table at line 29 and column 0. Hence the number 290 corresponds to the mantissa .4624. Put the decimal point as is indicated by the characteristic, and we have 2900.

Hence the required number for the logarithm 3.4624 is 2900.

(2) Given the logarithm 2.5950, find the corresponding number.

.5950 is not found in the table.

The adjacent mantissas are .5944 and .5955.

.5944 corresponds to 393.

.5955 corresponds to 394.

The difference between the mantissas is .0011, and the difference between .5944 and .5950 is .0006. Hence $\frac{6}{11}$ of the difference of the numbers corresponding to .5944 and .5955 must be added to the number corresponding to .5944.

The difference of the numbers is 1.

Hence $393 + \frac{6}{11}$ of $1 =$ number corresponding to $.5950 = 393.5.$

Hence the number corresponding to the given logarithm

$2.5950 = 393.5.$

(3) Given the logarithm $8.6474 - 10$, find the corresponding number.

.6474 is found in the table in line 44, column 4. Hence the corresponding number is 444. The characteristic is -2. Pointing off the number as is indicated by -2, we have .0444.

Hence the number corresponding to the logarithm

$$8.6474 - 10 = .0444.$$

407. The *cologarithm* of a number is the *arithmetical complement* of the logarithm of that number, and is the logarithm of the reciprocal of that number.

Thus, the cologarithm of $642 =$ logarithm of $\frac{1}{642} = \log 1 - \log 642$.

$$\begin{array}{lrl} \text{Thus,} & \log 1 = & 10. \quad -10 \\ & \log 642 = & \underline{2.8075 \qquad\ } \\ \text{Hence, the required cologarithm is} & & 7.1925 - 10 \end{array}$$

408. The process of finding the cologarithm may be simplified by beginning at the left and subtracting each figure of the logarithm from 9 except the last significant one, which subtract from 10. The remainder will be the cologarithm.

From the above it will be seen that the cologarithm of a number is the logarithm with a minus sign prefixed to it. This minus sign affects both characteristic and mantissa.

In order, therefore, to avoid a negative mantissa, the device of subtracting the logarithm from $10 - 10$, instead of from 0, will be found convenient.

APPLICATION OF LOGARITHMS.

409. The real or approximate value of an arithmetical quantity may be found by means of logarithms.

The process of working arithmetical quantities with logarithms is made simple from the fact that addition, subtraction, multiplication, and division take the place respectively of multiplication, division, involution, and evolution.

Example:

1. Find the product of $639 \times 544 \times .084$.

$$\log (639 \times 544 \times .084) = \log 639 + \log 544 + \log .084.$$

$$\begin{aligned} \log 639 &= 2.8055 \\ \log 544 &= 2.7356 \\ \log .084 &= \underline{8.9243 - 10} \\ \text{Adding,} \qquad &= 14.4654 - 10 \\ &= 4.4654. \end{aligned}$$

The number corresponding to $\log 4.4654 = 29200$.

The above must be taken only as the approximate answer. It differs from the actual answer by less than .3. Were we to use a six place table, the answer would be correct to three decimal places.

EXERCISE 121.

Find the logarithms of the following:

1. 986.42.
2. 3457.8.
3. 5.7852.
4. .00342.
5. 36 789.
6. 78921.
7. .06042.
8. .845464.
9. .00081.
10. .90001.
11. .47002.

EXERCISE 122.

Find the numbers corresponding to the following logarithms:

1. 1.9547.
2. $8.9934 - 10$.
3. 5.9996.
4. $1.8768 - 10$.
5. .0000.
6. 1.0253.
7. $7.8435 - 10$.
8. $9.7015 - 10$.
9. 2.8925.
10. 3.8486.
11. 1.9119.

EXERCISE 123.

Applying Arts. 390, 391, 392, 393, solve the following examples by means of logarithms:

1. 93.8×43.6.
2. $672 \times .0044$.
3. 549.6×78.49.
4. $32.5 \div 0.043$.
5. $844 \times (-.00381)$.
6. $-563 \times (-89.61)$.
7. $3.07 \div .0219$.
8. $-36.42 \div (-8.967)$.

9. 34^3.

10. 92.6^4.

11. 87.1^5.

12. $(\frac{19}{17})^3$.

13. $(\frac{111}{217})^5$.

14. $(\frac{81}{139})^{\frac{3}{4}}$.

15. $\sqrt[3]{346.78}$.

16. $\sqrt{\dfrac{342.5}{6789}}$.

17. $\sqrt[4]{\dfrac{15.64 \times 19.76}{17.28 \times 67.2}}$.

18. $\sqrt[6]{\dfrac{8962 \div 7432}{6145 \times 9782}}$.

19. $\sqrt[3]{\dfrac{3.57 \times 89.6 \times .0045}{.034 \times 6.721 \times 98.2}}$.

20. $\left\{\dfrac{36.78 \times .00921 \times 678.9}{897.2 \times .00756 \times .07439}\right\}^{\frac{3}{5}}$.

21. $\left\{\dfrac{36.7 \times 48.9 \times 6.72}{892^{\frac{1}{2}} \times 67.2^{\frac{1}{3}} \times 8.76}\right\}^{-\frac{2}{3}}$.

EXPONENTIAL EQUATIONS.

410. An Exponential Equation is one in which the exponent involves the unknown quantity. Thus, $a^x = b$, in which a and b are real positive quantities and x is unknown.

This class of equations is easily solved by means of logarithms.

EXAMPLES:

1. Solve $95^x = 32$.

By logarithms, we have,

$$\log (95^x) = \log 32.$$

$$\therefore \log 95 \times x = \log 32.$$

$$\therefore x = \frac{\log 32}{\log 95}$$

$$= \frac{1.5051}{1.9777}$$

$$= .761+.$$

EXERCISE 124.

Solve the following equations:

1. $12^x = 13.$

2. $19^x = 129.$

3. $14.5^x = 156.$

4. $.081^x = .36.$

5. $2.45^x = 672.$

Solve the following examples in Geometrical Progression:

6. Given a, r, and s. Find n (Art. 338).

7. Given r, l, and s. Find n.

8. Given a, l, s. Find n.

9. Given a, l, r. Find n.

10. $s = 15$, $a = 1$, $r = 2$. Find n.

11. $a = 1$, $l = 27$, $r = 3$. Find n.

12. $l = 128$, $r = 4$, $a = 2$. Find n.

CHAPTER XXXIV.

PERMUTATIONS AND COMBINATIONS.

411. Each arrangement which can be made by taking some or all of a number of things is called a **Permutation.**

412. Each selection which can be made by taking some or all of a number of things is called a **Combination.**

Thus, from the letters *a, b, c, d,* taken two at a time, twelve permutations can be made; namely, *ab, ac, ad, bc, bd, cd, ba, ca, da, cb, db, dc.*

From these same letters, *a, b, c, d,* taken two at a time, six combinations can be made; namely, *ab, ac, ad, bc, bd, cd.*

413. From the above it readily appears that in making *combinations* we are concerned only with the number of things each selection contains, but in forming *permutations* we must consider the order of these things.

Thus, from the letters *a, b, c,* taken all together, we obtain *one combination,* namely, *abc;* and *six permutations,* namely, *abc, acb, bca, bac, cab, cba.*

414. I. *If one operation can be performed in a ways, and when performed a second operation can be performed in b ways, then the two operations can be performed in* $a \times b$ *ways.*

For with any operation of the first we can associate any of the *b* operations of the second, so that, considering only one way of performing the first operation, we shall have *b* ways of performing both; but as there are *a* ways of performing the first, and each of these may be performed in b ways, we shall have in all $a \times b$ ways of performing the two operations.

Examples:

1. In a box there are four black balls and three white balls. In how many ways may two balls, one black and one white, be selected?

Now a black ball may be selected in four different ways, and a white ball in three different ways, and as any black ball may be put with any white ball, we shall have 4×3, or 12 different ways.

2. If there are five ferry boats plying between New York and Staten Island, in how many ways could a person make the round trip, not taking the same boat in going and returning?

Now there are five ways of going from New York to Staten Island, and with each way there are four ways for returning; therefore, the number of ways of making the round trip would be $5 \times 4 = 20$.

The above principle may be extended to three or more operations, each of which can be performed in a given number of ways.

That is, if one thing can be done in a ways, a second in b ways, a third in c ways, and a fourth in d ways, they can be performed together in $a \times b \times c \times d$ ways.

415. II. To find the number of permutations of n dissimilar things taken all together.

The first place can be filled in n ways, the second in $(n-1)$ ways, the third in $(n-2)$ ways, and so on to the last place, which can be filled in 1 way.

Hence, the formula:

$$n(n-1)(n-2)(n-3)\cdots 3 \times 2 \times 1, \qquad (1)$$

which is written $\underline{|n}$, and called "*factorial n.*"

EXAMPLE:

(1) How many permutations can be formed with the letters of the word *courage.*

There are seven letters.

The first place can be filled with any one of these seven letters. Hence, in seven ways.

The second place can be filled with any one of the remaining six letters. Hence, in six ways, and

The third place in five ways.
The fourth place in four ways.
The fifth place in three ways.
The sixth place in two ways.
The seventh place in one way.

Hence, $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$, or $\underline{|7}$ ways.

Multiplying, we have 5040, which are the possible permutations.

416. III. **To find the permutations of n dissimilar things taken r at a time.**

The first place can be filled in n ways, the second in $n-1$ ways, and the rth place in $n-(r-1)$ ways.

Hence, $n(n-1)(n-2)(n-3)\cdots$ *to* r *factors times will give the possible permutations of* n *things taken* r *at a time.*

The formula may be written:

$$n(n-1)(n-2)(n-3)\cdots n-(r-1)$$
$$=n(n-1)(n-2)(n-3)\cdots(n-r+1). \qquad (2)$$

417. ${}^nP_r$ represents the number of permutations of n things taken r at a time. When $r=n$, we have ${}^nP_r = {}^nP_n = \underline{|n}.$

EXAMPLE:

(1) How many permutations can be made of the letters of the word *courage* taken five at a time?

Here, $n=7,\ r=5.$

Substituting in the formula, we have

$$\begin{aligned} &7(7-1)(7-2)(7-3)(7-4)\\ &=7\times6\times5\times4\times3\\ &=2520.\end{aligned}$$

418. IV. **To find the number of combinations of n dissimilar things taken r at a time.**

Each combination consists of a group of r dissimilar things.

Now, each group of r dissimilar things makes $\underline{|r}$ permutations.

Hence, *the permutations of n dissimilar things taken r at a time divided by factorial r will give the combinations of n dissimilar things taken r at a time.*

Let ${}^nC_r$ represent the number of combinations of n dissimilar things taken r at a time.

Then, $${}^nP_r \div \underline{|r} = {}^nC_r;$$

that is, $$\frac{n(n-1)(n-2)\cdots(n-r+1)}{\underline{|r}} = {}^nC_r.$$

Multiplying both numerator and denominator of this formula by $\underline{|n-r}$, we shall have

$${}^nC_r = \frac{n(n-1)(n-2)\cdots(n-r+1)\underline{|n-r}}{\underline{|r}\,\underline{|n-r}}. \qquad (3)$$

In this last the numerator is equal to the product of all the natural numbers from n to 1, and, therefore, is equal to $\underline{|n}$.

Hence, the simple formula:

$${}^nC_r = \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}. \qquad (4)$$

419. If $n=r$, we have $\frac{\underline{|n}}{\underline{|n}\,\underline{|0}} = \frac{1}{\underline{|0}}$, but ${}^nC_n = 1$. $\therefore\ \frac{1}{\underline{|0}} = 1$.

EXAMPLE:

(1) From 10 horses how many combinations of 5 can be made?

Applying the formula, we have,

$$\frac{\underline{|10}}{\underline{|5}\,\underline{|5}}$$

$$= \frac{10 \times 9 \times 8 \times 7 \times 6}{1 \times 2 \times 3 \times 4 \times 5}$$

$$= 252.$$

(2) From 10 horses how many combinations of 5 can be made when 2 are always included?

Since 2 are in each combination, we have only to select 3 from the remaining 8.

Hence, $${}^nC_r = {}^8C_3 = \frac{8 \times 7 \times 6}{1 \times 2 \times 3}$$

$$= 56.$$

(3) From 10 horses how many combinations of 5 can be made when 2 are always excluded?

Since 2 are always excluded, we have only to make our selections from the remaining 8.

Hence, $${}^nC_r = {}^8C_5 = \frac{8 \times 7 \times 6 \times 5 \times 4}{1 \times 2 \times 3 \times 4 \times 5}$$

$$= 56.$$

420. V. **To prove the number of combinations of *n* things taken *r* at a time is equal to the number of combinations taken *n* − *r* at a time.**

For, $${}^nC_{n-r} = \frac{\underline{|n}}{\underline{|n-r}\,\underline{|n-(n-r)}}$$

$$= \frac{\underline{|n}}{\underline{|n-r}\,\underline{|r}} \qquad (5)$$

$$= {}^nC_r.$$

421. ${}^nC_r$ and ${}^nC_{n-r}$ are called complementary combinations.

The above formula (5) is often useful in abridging arithmetical work required by formula (4).

EXAMPLE:

(1) Out of 100 sheep, in how many ways can 95 be chosen?

The formula
$$\begin{aligned} {}^{100}C_{95} &= {}^{100}C_{100-95} \\ &= {}^{100}C_5 \\ &= \frac{100 \times 99 \times 98 \times 97 \times 96}{1 \times 2 \times 3 \times 4 \times 5} \\ &= 15057504. \end{aligned}$$

422. VI. **To find what value of r will give the greatest number of combinations.**

The formula

$$\begin{aligned} {}^nC_r &= \frac{n(n-1)(n-2)(n-3)\cdots(n-r+1)}{1 \times 2 \times 3 \times 4 \times \cdots r} \\ &= \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \cdots \frac{(n-r+1)}{r}. \end{aligned}$$

It is evident that the greatest product is reached when $\frac{n-r+1}{r}$ is greater than 1.

When n is odd, the numerator and denominator of each factor will be alternately odd and even.

Hence, *the factor greater than and nearest to* 1 *will have a numerator which exceeds the denominator by* 2, *and* $n-r+1$ *will equal* $r+2$, and *if* $r = \frac{n-1}{2}$, *we shall have the greatest number of combinations when n is odd.*

Hence, $${}^nC_{\frac{n-1}{2}}. \tag{6}$$

When n is even, the numerator of each factor will be alternately even and odd, and the denominator odd and even.

Hence, *the factor greater than and nearest to* 1 *will have a numerator which exceeds the denominator by* 1, *and* $n-r+1=r+1$, and *if* $r = \frac{n}{2}$, *we shall have the greatest number of combinations when n is even.*

Hence, $${}^nC_{\frac{n}{2}}. \qquad (7)$$

EXAMPLES:

(1) What value of r will give the greatest number of combinations out of 11 things?

The formula gives
$$\begin{aligned} r &= \frac{n-1}{2} \\ &= \frac{11-1}{2} \\ &= 5. \end{aligned}$$

$$\begin{aligned} \therefore\ {}^nC_{\frac{n-1}{2}} &= \frac{11 \times 10 \times 9 \times 8 \times 7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \\ &= 462. \end{aligned}$$

We have, also,
$$\begin{aligned} {}^nC_r = {}^{11}C_6 &= \frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 . 6} \\ &= 462. \end{aligned}$$

When n is odd the middle factor will equal 1, which, of necessity, will not change the result given by the preceding set of factors.

Hence, when n is odd, r may have two values, $\frac{n-1}{2}$ and $\frac{n-1}{2}+1$, or $\frac{n+1}{2}$.

(2) What value of r will give the greatest number of combinations out of 10 things?

The formula gives $$r = \frac{n}{2}.$$

$$\therefore\ r = \tfrac{10}{2} = 5.$$

$$\begin{aligned} \therefore\ {}^nC_{\frac{n}{2}} = {}^{10}C_5 &= \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{1 . 2 \cdot 3 \cdot 4 \cdot 5} \\ &= 252. \end{aligned}$$

423. VII. **To find the number of ways in which $m + n$ things, all different, can be divided into two groups containing respectively m and n things.**

This is equivalent to finding the combinations of m things from $m + n$ things.

Hence the formula,

$$ {}^{m+n}C_m = \frac{\underline{|m+n}}{\underline{|m}\,\underline{|n}}. \tag{8}$$

424. VIII. **To find the number of ways into which $m + n + p$ things can be divided into three groups containing, respectively, m, n, and p things.**

First divide into two groups m and $n + p$, respectively, and we have, by above,

$$ {}^{m+n+p}C_m = \frac{\underline{|m+n+p}}{\underline{|m}\,\underline{|n+p}}; $$

but $n + p$ can be divided into two groups containing n and p things respectively, giving, as above,

$$ \frac{\underline{|n+p}}{\underline{|n}\,\underline{|p}}. $$

$$ \therefore \frac{\underline{|n+n+p}}{\underline{|m}\,\underline{|n+p}} \times \frac{\underline{|n+p}}{\underline{|n}\,\underline{|p}} = \frac{\underline{|m+n+p}}{\underline{|m}\,\underline{|n}\,\underline{|p}}. \tag{9}$$

Now if in formula (8) $m = n$, we have

$$ \frac{\underline{|2m}}{\underline{|m}\,\underline{|m}}, $$

but as these groups may be interchanged one with the other, the number of ways will be

$$ \frac{\underline{|2m}}{\underline{|m}\,\underline{|m}\,\underline{|2}}. $$

Likewise in formula (9), if $m = n = p$, we shall have

$$\frac{\underline{|3m}}{\underline{|m}\,\underline{|m}\,\underline{|m}\,\underline{|3}}.$$

EXAMPLE:

(1) Find the number of ways in which 15 Republicans can be formed in 3 committees of 8, 5, and 2, respectively.

The formula gives $\dfrac{\underline{|15}}{\underline{|8}\,\underline{|5}\,\underline{|2}} = 335135.$

Into 3 groups of 5 each.

The formula gives $\dfrac{\underline{|15}}{\underline{|5}\,\underline{|5}\,\underline{|5}\,\underline{|3}} = 17918.$

425. IX. **To find the permutations of n things, all at a time, when there are p things of one kind, q of another, and r of another.**

Let X be the total number of permutations.

Now if in any one of these permutations p unlike things replaced p like things, the number of permutations would be increased by $\underline{|p}$ without altering position of q and r, and if this change were made in X permutations, we should have $X \times \underline{|p}$ permutations. So, also, if q and r unlike things replaced q and r like things, we should have $X \times \underline{|p} \times \underline{|q} \times \underline{|r} = \underline{|n} =$ the number of permutations of n things all different.

$$\therefore\ X = \frac{\underline{|n}}{\underline{|p}\,\underline{|q}\,\underline{|r}}, \tag{10}$$

the total number of permutations.

EXAMPLE:

(1) How many permutations can be made with the letters of the word *Mississippi?*

We have 11 letters as follows: 1 m, 4 i's, 4 s's, 2 p's.

$$\therefore\ \frac{\underline{|11}}{\underline{|1}\,\underline{|4}\,\underline{|4}\,\underline{|2}} =$$

the total number of permutations $= 34650$.

426. X. **To find the number of permutations of n things, r at a time, when each thing may be repeated 1 to r times in any permutation.**

Consider the ways in which r places can be filled. The first place can be filled in n ways, and since repetitions are allowed, the second place can be filled in n ways. Therefore, the first two places can be filled in n^2 ways. The first three places (since the third place can be filled in n ways) in n^3 ways, and in like manner, the first r places in n^r.

Example:

(1) If any combination of 3 letters make a word, how many words of 3 letters each can be made from 10 letters?

The first place can be filled in 10 ways, the second place in 10 ways, and the third place in 10 ways.

$$\therefore\ 10^3 = \text{the number of words, which is } 1000.$$

427. XI. **To find the total number of combinations which can be made of some or all of n things.**

Each thing can be taken or rejected. Hence, each of the n things can be disposed of in two ways, or 2^n ways. In one case, however, both are rejected. Hence, *the total number of ways is*

$$2^n - 1. \tag{11}$$

Example:

(1) How many different sums can be made with five different coins?

The formula gives $2^5 - 1 = 31$.

428. XII. **To find the total number of combinations of $p + q + r + \cdots$ things, of which p are alike, q are alike, r are alike, and so on, when some or all are taken.**

Now p things may be disposed of in $p + 1$ ways, since we may take none or any of them.

Also, q things may be disposed of in $q+1$ ways, and r things in $r+1$ ways, and so on.

Hence, the number of ways in which all may be disposed of is

$$(p+1)(q+1)(r+1)\cdots.$$

This includes the case in which all are rejected, which is inadmissible.

Hence, the number of ways is

$$(p+1)(q+1)(r+1)\cdots-1. \tag{12}$$

EXAMPLE:

(1) In how many ways can we select, one or all, from 5 arithmetics, 3 readers, and 4 spellers?

The arithmetics may be disposed of in $(p+1)$ ways, or 6 ways.

The readers in $(q+1)$, or 4 ways.

The spellers in $(r+1)$, or 5 ways.

Hence, $6\cdot 4\cdot 5-1=119.$

429. XIII. **Permutations and combinations of *n* things, not all different, taken *r* at a time.**

EXAMPLES:

(1) How many combinations can be made from 6 flags, 4 white and 2 black, taken 4 at a time?

(2) How many permutations?

Combinations:

(1) 4 white,

(2) 3 white, 1 black,

(3) 2 white, 2 black.

Hence, 3 combinations.

Permutations:

From (1) we have $\left\{\frac{|\underline{4}}{|\underline{4}}\right. = 1,$

From (2) we have $\left\{\frac{|\underline{4}}{|\underline{3}\,|\underline{1}}\right. = 4,$

From (3) we have $\left\{\frac{|\underline{4}}{|\underline{2}\,|\underline{2}}\right. = 6.$

Hence, 11 permutations.

EXERCISE 125.

1. How many permutations can be made with the letters of the word *Constantinople?*

2. How many words, consisting of 4 consonants and 2 vowels, can be formed from 10 consonants and 5 vowels?

3. There are 7 candidates for a classical, 5 for a mathematical, and 4 for a divinity scholarship, in how many ways can the scholarships be awarded?

4. How many permutations can be made from the letters in the word *Ecclesiastical?*

5. With 4 consonants and 3 vowels, how many words of 5 letters can be formed?

6. With 6 consonants and 4 vowels, how many words of 4 letters each can be formed, if each begins with a consonant and ends with a vowel?

7. Out of 30 soldiers, in how many ways can 6 officers be chosen, and how many times would any man be chosen?

8. How many words of 5 letters can be made from the letters of the word *triangle,* if all end with the letter e?

9. Find the value of ${}^{12}P_4$, ${}^{12}C_4$.

10. Find the number of combinations of 75 things taken 70 at a time.

11. How many changes can be rung with 7 bells?

12. In how many ways can a football eleven be chosen from 39 men, the captain always being one?

13. Out of 25 candidates for a baseball nine, in how many ways can the three bases be filled?

14. How many different throws can be made with five dice, if a six always turns up?

15. How many ways can the men of an eight-oared crew be arranged in the boat, if the captain always rows stroke?

16. At a whist table, what are the chances that *A* will have all clubs, *B* all spades, *C* all hearts, *D* all diamonds? Give formula only.

17. Out of 20 consonants and 5 vowels, how many words can be formed consisting of 4 consonants and 3 vowels?

18. How many different committees can be formed from 16 Republicans and 10 Democrats, if each committee consists of 4 Republicans and 3 Democrats?

19. How many ways can a crew of eight be arranged, if two always row on starboard and one on port?

20. How many signals can be made with eight flags of different colour, four at a time?

21. How many different signals can be made with 10 lights of different colour, 3 at a time?

22. How many signals can be made with 12 flags, of which 6 are red, 3 are white, 2 are blue, and 1 is black, if they may be hoisted one above the other, side by side, or diagonally?

23. Out of 10 physicians, 8 lawyers, and 12 ministers, how many committees can be formed consisting of 3 physicians, 2 lawyers, and 4 ministers?

24. In how many ways can 25 tutors be assigned to watch 3 examinations, 3 to the first, 4 to the second, and 5 to the third?

CHAPTER XXXV.

INTEREST AND ANNUITIES.

430. I. **To find the simple interest or amount of a given sum for a given time.**

431. The **Principal** is the sum lent.

The **Interest** is the money paid for what is lent.

The **Amount** is the sum of the principal and interest at the time the money is due.

Simple Interest is paid on the principal alone.

Let P represent the principal.
r the interest on $1 for 1 year.
n the number of years.
R the amount of $1 for 1 year.
A the amount of P for n years.

Hence,

$$R = 1 + r.$$
$$Pr = \text{Interest on } P \text{ for 1 year.}$$
$$Pnr = \text{Interest on } P \text{ for } n \text{ years.}$$
$$P(1 + nr) = \text{Amount of } P \text{ for } n \text{ years.}$$

That is,

$$P(1 + nr) = A. \qquad (1)$$

When *interest* is mentioned, simple interest is meant.

When three of the numbers P, n, r, A are given, the fourth can be found.

EXAMPLE:

Find the number of years in which $250 will amount to $500 at 6%.

Here A, P, and n are given, and r required.

$$A = P(1 + nr).$$
$$A = P + Pnr.$$
$$Pnr = A - P.$$
$$n = \frac{A - P}{Pr}.$$
$$\therefore\ n = \frac{500 - 250}{15.00} = 16\tfrac{2}{3} \text{ years.}$$

432. II. **To find the interest or amount of a given sum at compound interest for a given time.**

(I.) **When payable annually.**

$$P(1 + r) \text{ or } PR = \text{Amount for 1 year.}$$
$$PR(1 + r) \text{ or } PR^2 = \text{Amount for 2 years.}$$
$$PR^{n-1}(1 + r) \text{ or } PR^n = \text{Amount for } n \text{ years.}$$

That is, $$A = PR^n. \tag{2}$$

433. (II.) **When payable semiannually.**

$$P\left(1 + \frac{r}{2}\right) = \text{Amount for } \tfrac{1}{2} \text{ year.}$$
$$P\left(1 + \frac{r}{2}\right)^2 = \text{Amount for 1 year.}$$
$$P\left(1 + \frac{r}{2}\right)^{2n} = \text{Amount for } n \text{ years.}$$

That is, $$A = P\left(1 + \frac{r}{2}\right)^{2n}. \tag{3}$$

In like manner we may show that when the interest is payable *quarterly* or *monthly*, that

$$A = P\left(1 + \frac{r}{4}\right)^{4n} \tag{4}$$

and $$A = P\left(1 + \frac{r}{12}\right)^{12n}, \tag{5}$$

respectively.

Examples:

(1) Find the compound interest of \$ 525 for 1 year 6 months, at 5%, payable quarterly.

Here Pr and n are given and A required.

$$A = P\left(1 + \frac{r}{4}\right)^{4n}.$$

Performing the operation indicated by means of logarithms, we have:

$$A = \$\,525\left(1 + \frac{.05}{4}\right)^{6} = \$\,565.62.$$

Hence, compound interest $= \$\,40.62$.

(2) Find the present value of \$ 1000, due in 5 years, at 10% compound interest.

$$A = P(1 + r)^5.$$

$$P = \frac{A}{(1 + r)^5}$$

$$= \frac{\$\,1000}{1.10^5} = \$\,621 \text{ nearly}.$$

Annuities. An annuity is a fixed sum of money to be paid under stated conditions at equal intervals of time.

It may be:

Certain, *i.e.* payable independent of any condition.
Life, *i.e.* payable during the life of some certain individual.
Deferred, *i.e.* payable after the lapse of a number of years.
Perpetual, *i.e.* payable forever.

434. III. **To find the present worth of an annuity for a number of years, allowing compound interest.**

435. Let A = Annuity in dollars.
R = Amount of \$ 1 for 1 year.
P = Present worth of annuity.

By Art. (432), annual payments,

$$A = PR \text{ at end of 1st year.}$$
$$A = PR^2 \text{ at end of 2d year.}$$
$$A = PR^n \text{ at end of } n\text{th year.}$$
$$\therefore\ P = \frac{A}{R} \text{ at end of 1st year.}$$
$$P = \frac{A}{R^2} \text{ at end of 2d year.}$$
$$P = \frac{A}{R^n} \text{ at end of } n\text{th year.}$$

Hence the present worth of the annuity is

$$\frac{A}{R^n} + \frac{A}{R^{n-1}} + \frac{A}{R^{n-2}} + \cdots + \frac{A}{R^2} + \frac{A}{R}.$$

That is, $$P = A\left[\frac{1}{R^n} + \frac{1}{R^{n-1}} + \frac{1}{R^{n-2}} + \cdots + \frac{1}{R^2} + \frac{1}{R}\right].$$

$\therefore$ By Art. (335), $$P = A\frac{1 - \frac{1}{R^n}}{R-1}. \qquad (6)$$

EXAMPLE:

(1) Find the present worth of an annuity of \$ 500 to continue for 15 years, with compound interest at 4 per cent.

Here, $A = 500,\ n = 15,\ R = 1.04,\ R - 1 = .04,$

whence, $$P = \frac{500}{.04}\left[1 - \frac{1}{(1.04)^{15}}\right],$$

$$\begin{array}{rr} \text{colog } 1.04 = & 9.9830 - 10 \\ & 15 \\ \hline & 149.7450 - 150 \\ & 9.7450 - 10 \end{array}$$

Number corresponding to log 9.7450 − 10 = .5559.

Hence, $$P = \frac{500}{.04}(1 - .5559)$$
$$= \$\,5551.25.$$

436. Formula (6) gives

$$A=\frac{P(R-1)}{1-\frac{1}{R^n}}=\frac{PR^n(R-1)}{R^n-1}, \tag{7}$$

which applies when the present worth and the amount of one dollar for one year are given.

If in formula (6) n is increased indefinitely, the limiting value of the second member becomes

$$\frac{A}{R-1}. \tag{8}$$

That is, *the present worth of a perpetual annuity is equal to the amount divided by the rate.*

From formula (6) we again have

$$\frac{A(R^n-1)}{R^n(R-1)}. \tag{9}$$

By Art. (435) the present worth of the above amount due in p years would be

$$\frac{A(R^n-1)}{R^n(R-1)}\div R^p.$$

Hence,

$$P=\frac{A(R^n-1)}{R^{p+n}(R-1)}, \tag{10}$$

which applies when the present worth of an annuity is required, which begins after p years and continues for n years thereafter.

From formulas (10) and (8) we obtain

$$P=\frac{A}{R^p(R-1)}, \tag{11}$$

which applies when the present worth of an annuity is required, which begins after p years and continues in perpetuity.

EXERCISE 126.

1. Find the interest of $350 for 10 years at 7 per cent, simple interest.

2. Find the compound interest of $1000 for 6 years at 5 per cent.

3. What will be the amount of $1200 at 8 per cent for 10 years, compound interest?

4. What sum of money will amount to $1050 in 4 years at 3 per cent, compound interest?

5. What is the present worth of a note of $1580, due in 12 years at 5 per cent, interest compounded semiannually?

6. At what rate per cent yearly will $1850 amount to $2275 in 4 years and 6 months, interest compounded quarterly?

7. In how many years will $200 double itself at 6 per cent, annual compound interest?

8. In how many years will a sum of money quadruple itself at 10 per cent, semiannual compound interest?

9. What is the present worth of an annuity of $400, to continue 20 years at 7 per cent, compound interest?

10. What is the present worth of an annuity of $1250, to continue at 4 per cent for 20 years, interest compounded annually?

11. What is the present worth of a perpetual annuity of $150, to begin after 8 years, annual compound interest at 5 per cent?

12. What annuity can be purchased for $1284.70, if it is to run for 14 years at 6 per cent, annual compound interest?

CHAPTER XXXVI.

CONTINUED FRACTIONS.

437. A **Continued Fraction** is an expression of the following form:

$$a+\cfrac{b}{c+\cfrac{d}{e+\cfrac{b}{g+}}}.$$

It is usually written

$$a+\frac{b}{c+}\,\frac{d}{e+}\,\frac{b}{g+}.$$

438. Any proper fraction in its lowest terms may be converted into a continued fraction by the method of finding the G. C. D.

439. If the number of the denominators of the fraction is finite, it is called a **Terminating** continued fraction.

440. If the number of denominators of the fraction is indefinitely great, it is called an **Infinite** continued fraction.

Convergents. In the continued fraction

$$a_1+\frac{1}{a_2+}\,\frac{1}{a_3}\,\frac{1}{a_4+}\cdots$$

a_1 is called the *first convergent* $a_1+\frac{1}{a_2}$ is called the *second convergent,* $a_1+\frac{1}{a_2}+\frac{1}{a_3}$ is called the *third convergent,* and so on.

Let $\frac{a}{b}$ be any fraction in its lowest terms.

Divide a by b, and let a_1 equal the quotient and b_1 the remainder, then

$$\frac{a}{b}=a_1+\frac{b_1}{b}=a_1+\cfrac{1}{\cfrac{b}{b_1}}.$$

Now, dividing b by b_1, and letting a_2 equal the quotient and b_2 the remainder, we have,

$$\frac{a}{b}=a_1+\cfrac{1}{a_2+\cfrac{b_2}{b_1}}=a_1+\cfrac{1}{a_2+\cfrac{1}{\cfrac{b_1}{b_2}}},$$

and, again, by like process,

$$\frac{a}{b}=a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\cfrac{1}{\cfrac{b_2}{b_3}}}}.$$

441. This process, which corresponds to that of finding the H. C. D., must finally give unity for a remainder, therefore the fraction $\frac{a}{b}$ may be, a and b being prime to each other, converted into a *terminating continued fraction.*

EXAMPLE:

(1) Convert $\frac{743}{611}$ into a continued fraction.

```
611)743(1
    611
    132)611(4                4)15(3
        528                    12
         83)132(1               3)4(1
             83                   3
             49)83(1              1)3(3
                49
                34)49(1
                   34
                   15)34(2
                      30
```

Therefore $a_1 = 1$, $a_2 = 4$, $a_3 = 1$, $a_4 = 1$, $a_5 = 1$, $a_6 = 2$, $a_7 = 3$, $a_8 = 1$, $a_9 = 3$.

Hence, the continued fraction:

$$\frac{743}{611} = 1 + \frac{1}{4+}\;\frac{1}{1+}\;\frac{1}{1+}\;\frac{1}{1+}\;\frac{1}{2+}\;\frac{1}{3+}\;\frac{1}{1+}\;\frac{1}{3}.$$

442. I. **To convert a quadratic surd into an infinite continued fraction.**

$$(1)\quad \sqrt{6} = 2 + \sqrt{6} - 2 = 2 + \frac{2}{\sqrt{6}+2} = 2 + \frac{1}{\frac{\sqrt{6}+2}{2}};$$

$$(2)\quad \frac{\sqrt{6}+2}{2} = 2 + \frac{\sqrt{6}-2}{2} = 2 + \frac{1}{\sqrt{6}+2} = 2 + \frac{1}{\frac{\sqrt{6}+2}{1}};$$

$$(3)\quad \frac{\sqrt{6}+2}{1} = 4 + \frac{\sqrt{6}-2}{1} = 4 + \frac{2}{\sqrt{6}+2} = 4 + \frac{1}{\frac{\sqrt{6}+2}{2}};$$

and we have, as the steps now recur or repeat,

$$\sqrt{6} = 2 + \frac{1}{2+}\;\frac{1}{4+}\;\frac{1}{2+}\;\frac{1}{4+}, \text{ etc.}$$

In the above process the expressions at the beginning of the lines (1), (2), (3) are separated into two parts; the first part being the greatest integer the expression contains, and the second part being the remainder.

The greatest integer in $\sqrt{6}$ is 2.

The greatest integer in $\frac{\sqrt{6}+2}{2}$ is 2.

The operation is as follows:

$$\sqrt{6} = 2 + \sqrt{6} - 2 = 2 + \frac{(\sqrt{6}-2)(\sqrt{6}+2)}{\sqrt{6}+2}$$

$$= 2 + \frac{2}{\sqrt{6}+2} = 2 + \frac{1}{\frac{\sqrt{6}+2}{2}},$$

and again,

$$\frac{\sqrt{6}+2}{2}=2+\frac{\sqrt{6}-2}{2}$$

$$=2+\frac{(\sqrt{6}-2)(\sqrt{6}+2)}{2(\sqrt{6}+2)}=2+\frac{1}{\sqrt{6}+2}.$$

The remainder in each case is made to have a rational numerator, and is expressed as a fraction with unity as a numerator.

The quotients begin to repeat themselves when we arrive at a quotient which is double the first. This is always the case.

A repeating infinite continued fraction is called a **Periodic Fraction.**

443. II. **A periodic continued fraction may be expressed as the root of a quadratic surd.**

EXAMPLE:

(1) $$\frac{1}{1+}\,\frac{1}{4+}\,\frac{1}{1+}\,\frac{1}{4+}\cdots. \qquad (1)$$

Let x equal the value of the fraction:

Then, $$x=\frac{1}{1+}\,\frac{1}{4+x}=\frac{4+x}{4+x+1}=\frac{4+x}{5+x}.$$

Clearing of fractions, $5x+x^2=4+x$

$$=x^2+4x=4.$$

Solving, we have $$x=-2+2\sqrt{2}.$$

444. III. **The convergents are alternately less and greater than the continued fraction.**

Let the continued fraction be

$$a_1+\frac{1}{a_2+}\,\frac{1}{a_3+}\cdots.$$

The first convergent, a_1, is too small because the part $\frac{1}{a_2+}\,\frac{1}{a_3+}\cdots$ is omitted.

The second convergent, $a_1+\frac{1}{a_2}$, is too large because the denominator a_2 is too small.

The third convergent, $a_1+\frac{1}{a_2+}\,\frac{1}{a_3}$, is too small because $a_2+\frac{1}{a_3}$ is too great, and so on.

If the given continued fraction is a proper fraction, $a_1 \doteq 0$. Therefore we may, in this case, consider the first convergent equal to zero.

445. Hence, *the odd convergents are less, and the even convergents greater than the continued fraction.*

446. IV. **To show the relation of the successive convergents.**

Let the continued fraction be

$$a_1+\frac{1}{a_2+}\,\frac{1}{a_3+}\,\frac{1}{a_4+}\cdots.$$

The first three convergents when reduced are

$$\frac{a_1}{1},\ \frac{a_1a_2+1}{a_2},\ \frac{a_3(a_1a_2+1)+a_1}{a_3a_2+1}.$$

An examination of the above shows that the numerator of the third convergent is formed by multiplying the numerator of the second convergent by the last denominator taken, and adding the numerator of the first convergent. Also, that the numerator of the second convergent is formed in a similar way. Also, that the denominator of the third convergent is formed by multiplying the denominator of the second convergent by the last denominator taken, and adding the denominator of the first convergent.

Let the numerators of the successive convergents be denoted by $p_1, p_2, p_3, \cdots$, and the denominators by $q_1, q_2\, q_3, \cdots$.

Assume the law holds for the nth convergent, that is,

$$p_n = a_n p_{n-1} + p_{n-2}$$

and

$$q_n = a_n q_{n-1} + q_{n-2}.$$

$$\therefore \frac{p_n}{q_n} = \frac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}}.$$

447. The $(n+1)$th convergent differs from the nth in having $a_n + \frac{1}{a_{n+1}}$ instead of a_n.

Hence, the $(n+1)$th convergent will be as follows:

$$\frac{p_{n+1}}{q_{n+1}} = \frac{\left(a_n + \frac{1}{a_{n+1}}\right)p_{n-1} + p_{n-2}}{\left(a_n + \frac{1}{a_{n+1}}\right)q_{n-1} + q_{n-2}}$$

$$= \frac{a_{n+1}(a_n p_{n-1} + p_{n-2}) + p_{n-1}}{a_{n+1}(a_n q_{n-1} + q_{n-2}) + q_{n-1}} = \frac{a_{n+1}p_n + p_{n-1}}{a_{n+1}q_n + q_{n-1}}.$$

Hence, the numerator and denominator of the $(n+1)$th convergent follow the law which was assumed for the nth convergent. But as the law holds in case of the third, it holds in case of the fourth, and so on to the nth and $(n+1)$th case. Therefore the law is universal.

448. V. **The difference between any two consecutive convergents, $\frac{p_n}{q_n}$ and $\frac{p_{n+1}}{q_{n+1}}$, equals $\frac{1}{q_n q_{n+1}}$.**

$$\left(a_1 + \frac{1}{a_2}\right) - a_1 = \frac{1}{a_2}.$$

Hence, the law holds for the first and second convergents.

Assume that it holds for the nth and $(n+1)$th convergents. That is,

$$\frac{p_n}{q_n} \sim \frac{p_{n+1}}{q_{n+1}} = \frac{1}{q_n q_{n+1}} \text{ or } p_n q_{n+1} \sim p_{n+1} q_n = 1.$$

Then, $$\frac{p_{n+1}}{q_{n+1}} \sim \frac{p_{n+2}}{q_{n+2}} = \frac{p_{n+1}}{q_{n+1}} \sim \frac{a_{n+2}p_{n+1} + p_n}{a_{n+2}q_{n+1} + q_n}$$

$$= \frac{(a_{n+2}p_{n+1}q_{n+1} + p_{n+1}q_n) \sim (a_{n+2}p_{n+1}q_{n+1} + p_n q_{n+1})}{q_{n+1}(a_{n+2}q_{n+1} + q_n)}$$

$$= \frac{p_{n+1}q_n \sim p_n q_{n+1}}{q_{n+1}q_{n+2}} = \frac{1}{q_{n+1}q_{n+2}}.$$

Hence, as the law holds for any pair of consecutive convergents, it is universal.

449. VI. **Each convergent is nearer than the preceding convergent to the value of the continued fraction.**

By Arts. 446–447, $\frac{p_{n+2}}{q_{n+2}}=\frac{a_{n+2}p_{n+1}+p_n}{a_{n+2}q_{n+1}+q_n}.$

Putting $$a_{n+2}+\frac{1}{a_{n+3}+\cdots}\text{ for }a_{n+2},$$

and denoting the value of the fraction by x, we have,

$$x=\frac{\left\{a_{n+2}+\frac{1}{a_{n+3}+\cdots}\right\}p_{n+1}+p_n}{\left\{a_{n+2}+\frac{1}{a_{n+3}+\cdots}\right\}q_{n+1}+q_n}=\frac{mp_{n+1}+p_n}{mq_{n+1}+q_n},$$

where m is substituted for $a_{n+2}+\frac{1}{a_{n+3}+\cdots}$.

Again, $$x\sim\frac{p_n}{q_n}=\frac{mp_{n+1}+p_n}{mq_{n+1}+q_n}\sim\frac{p_n}{q_n}$$

$$=\frac{m(p_{n+1}q_n\sim p_nq_{n+1})}{q_n(mq_{n+1}+q_n)}$$

$$=\frac{m}{q_n(mq_{n+1}+q_n)}. \qquad (1)$$

By like reasoning,

$$x\sim\frac{p_{n+1}}{q_{n+1}}=\frac{1}{q_{n+1}(mq_{n+1}+q_n)}. \qquad (2)$$

Now a_{n+2} is a positive integer.

Therefore, $$a_{n+2}+\frac{1}{a_{n+3}+\cdots}>1.$$

And since $q_{n+1}=a_{n+1}q_n+q_{n-1}$, $\quad q_{n+1}>q_n$.

Therefore (2) is less than (1).

Therefore the $(n+1)$th convergent is nearer than the nth convergent to the value of the fraction.

EXERCISE 127.

Calculate the successive convergents:

1. $3+\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{9}.$

2. $2+\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{5+}\,\frac{1}{1+}\,\frac{1}{5+}\,\frac{1}{1+}\cdots.$

Convert each of the following into continued fractions, and find the first four convergents:

1. $\frac{112}{157}$.
2. $\frac{73}{89}$.
3. $\frac{144}{61}$.
4. $\frac{3015}{6961}$.
5. $\frac{445}{612}$.
6. $\frac{729}{2818}$.
7. $\frac{967}{153}$.
8. $\frac{47}{159}$.

Convert each of the following to a continued fraction, and find the first four convergents:

1. $\sqrt{3}$.
2. $\sqrt{5}$.
3. $\sqrt{13}$.
4. $\sqrt{17}$.
5. $3\sqrt{2}$.
6. $\sqrt{14}$.
7. $\sqrt{43}$.
8. $\sqrt{\frac{7}{11}}$.
9. 1.093.
10. 3.1416.
11. .37.
12. 1.13.
13. .24226.
14. .76453.

Express as quadratic surds:

1. $3+\frac{1}{1+}\,\frac{1}{8+}\,\frac{1}{1+}\,\frac{1}{8+}\cdots.$

2. $\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{2+}\cdots.$

3. $1+\frac{1}{3+}\,\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{2+}\cdots.$

4. $\frac{1}{3+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{2+}\,\frac{1}{1+}\cdots.$